

A SPECTRAL SEQUENCE FOR POLYHEDRAL PRODUCTS

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This paper is dedicated to Samuel Gitler Hammer who brought us much joy and interest in Mathematics

ABSTRACT. The purpose of this paper is to exhibit fine structure for polyhedral products $Z(K; (\underline{X}, \underline{A}))$, and polyhedral smash products $\widehat{Z}(K; (\underline{X}, \underline{A}))$. Moment-angle complexes are special cases for which $(X, A) = (D^2, S^1)$. There are three main parts to this paper.

- (1) One part gives a natural filtration of the polyhedral product together with properties of the resulting spectral sequence in Theorem 2.15. Applications of this spectral sequence are given.
- (2) The second part uses the first to give a homological decomposition of $\widehat{Z}(K; (\underline{X}, \underline{A}))$ CW pairs $(\underline{X}, \underline{A})$.
- (3) Applications to the ring structure of $Z(K; (\underline{X}, \underline{A}))$ are given for CW-pairs (X, A) satisfying suitable freeness conditions.

1. Introduction

The subject of this paper is the homology of polyhedral products $Z(K; (\underline{X}, \underline{A}))$, and polyhedral smash products $\widehat{Z}(K; (\underline{X}, \underline{A}))$ [9, 5, 1, 2, 3]. Definitions are listed in section 2 of this paper.

One of the purposes of this article is to give the Hilbert-Poincaré series for the polyhedral product $Z(K; (\underline{X}, \underline{A}))$ in terms of

- (1) the kernel, image, and cokernel of the induced maps

$$H^*(X_i) \rightarrow H^*(A_i)$$

for all i , and

- (2) the full sub-complexes of K .

This computation was also worked out in [4] using more geometric methods.

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This is achieved by analysis of a spectral sequence abutting to the cohomology of the polyhedral product $Z(K; (\underline{X}, \underline{A}))$ by filtering this space with the left-lexicographical ordering of simplices. The method applies to a generalized multiplicative cohomology theory, h^* as well. The spectral sequence is then used to describe some features of the ring structure of $h^*(Z(K; (\underline{X}, \underline{A})))$.

Qibing Zheng [12] gives an alternative description of the cohomology of a polyhedral product. Our methods are distinct from his and the presentation of the computational results assumes a different form. Unlike the spectral sequence developed here, his collapses at the E_2 term.

2. Definitions, and main results

The basic constructions addressed in this article are defined in this section. First recall the definition of an abstract simplicial complex.

- Definition 2.1.** (1) Let K denote an abstract simplicial complex with m vertices labeled by the set $[m] = \{1, 2, \dots, m\}$. Thus, K is a subset of the power set of $[m]$ such that an element given by a $(k-1)$ -simplex σ of K is given by an ordered sequence $\sigma = (i_1, \dots, i_k)$ with $1 \leq i_1 < \dots < i_k \leq m$ such that if $\tau \subset \sigma$, then τ is a simplex of K . In particular the empty set \emptyset is a subset of σ and so it is in K . The vertex set of σ , $\{i_1, \dots, i_k\}$ will be denoted $|\sigma|$.
- (2) Given a sequence $I = (i_1, \dots, i_k)$ with $1 \leq i_1 < \dots < i_k \leq m$, define $K_I \subseteq K$ to be the full sub-complex of K consisting of all simplices of K which have all of their vertices in I , that is $K_I = \{\sigma \cap I \mid \sigma \in K\}$.
- (3) In case $I = (i_1, \dots, i_k)$, define $X^I = X_{i_1} \times X_{i_2} \times \dots \times X_{i_k}$.
- (4) Let $\Delta[m-1]$ denote the abstract simplicial complex given by the power set of $[m] = \{1, 2, \dots, m\}$.

Let h^* be a generalized, multiplicative cohomology theory and $(\underline{X}, \underline{A})$ denote a collection of based CW pairs $\{(X_i, A_i, x_i)\}_{i=1}^m$. We will also assume $h^*(X_i)$ and $h^*(A_i)$ are finite type, i.e. $h^*(X_i)$ and $h^*(A_i)$ are generated as h^* modules by classes, $\{x_\ell\}$ and $\{a_\ell\}$ respectively with finitely many generators in each degree.

For a generalized cohomology theory, h^* , we now describe a strong freeness condition on $(\underline{X}, \underline{A})$ that will be imposed in Section 4.

The strong freeness condition assumes that the long exact sequence

$$\xrightarrow{\delta} \widetilde{h}^*(X_i/A_i) \xrightarrow{\ell} h^*(X_i) \xrightarrow{\iota} h^*(A_i) \xrightarrow{\delta} \widetilde{h}^{*+1}(X_i/A_i) \rightarrow$$

can be written in terms of explicit, free h^* modules E_i, B_i, C_i and W_i .

Definition 2.2. *The pair $(\underline{X}, \underline{A})$ is said to satisfy a strong h^* freeness condition if there are free h^* -modules E_i, B_i, C_i and W_i satisfying*

$$(1) \ h^*(A_i) = E_i \oplus B_i \ (B_i \ni 1 \subset h^0(A_i)).$$

$$(2) \ h^*(X_i) = B_i \oplus C_i$$

$$\text{where } B_i \xrightarrow[\simeq]{\ell} B_i, \quad C_i \xrightarrow{\ell} 0$$

$$(3) \ \tilde{h}^*(X_i/A_i) = C_i \oplus W_i.$$

$$\text{where } C_i \xrightarrow[\simeq]{\ell} C_i, \quad E_i \xrightarrow[\simeq]{\delta} W_i \xrightarrow{\ell} 0$$

The goal of the spectral sequence is to compute the cohomology of the polyhedral product defined below. Our answer will be given in terms of the strong h^* free decomposition described in Definition 2.2. In particular the description of the cohomology is only natural with respect to mappings of $h^*(X_i)$ and $h^*(A_i)$ which preserve the chosen strong h^* decomposition. This point is further developed at the end of section 5.

In the following definition \mathcal{K} denotes the category of simplicial complexes and \mathcal{CW}_* is the category of based CW pairs.

Definition 2.3. (1) *The polyhedral product determined by $(\underline{X}, \underline{A})$ and K denoted*

$$Z(K; (\underline{X}, \underline{A}))$$

is defined using the functor

$$D : \mathcal{K} \rightarrow \mathcal{CW}_*$$

as follows: For every σ in K , let

$$D(\sigma) = \prod_{i=1}^m Y_i, \quad \text{where } Y_i = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \in [m] - \sigma \end{cases}$$

with $D(\emptyset) = A_1 \times \dots \times A_k$.

(2) *The polyhedral product is*

$$Z(K; (\underline{X}, \underline{A})) = \bigcup_{\sigma \in K} D(\sigma) = \text{colim} D(\sigma)$$

where the colimit is defined by the inclusions, $d_{\sigma, \tau}$ with $\sigma \subset \tau$ and $D(\sigma)$ is topologized as a subspace of the product $X_1 \times \dots \times X_k$. The polyhedral product is the underlying space $Z(K; (\underline{X}, \underline{A}))$ with base-point $\underline{} = (x_1, \dots, x_k) \in Z(K; (\underline{X}, \underline{A}))$.*

- (3) In the special case where $X_i = X$ and $A_i = A$ for all $1 \leq i \leq m$, it is convenient to denote the polyhedral product by $Z(K; (X, A))$ to coincide with the notation in [10].

A direct variation of the structure of the polyhedral product follows next. Spaces analogous to polyhedral products are given next where products of spaces are replaced by smash products, a setting in which non-degenerate base-points are required. We will always assume the pairs (X, A) are based CW pairs, in which case the base point condition is always satisfied.

The (reduced) suspension of a (pointed) space $(X, *)$

$$\Sigma(X)$$

is the smash product

$$S^1 \wedge X.$$

Definition 2.4. Given a polyhedral product $Z(K; (\underline{X}, \underline{A}))$ obtained from $(\underline{X}, \underline{A}, *)$, the polyhedral smash product

$$\widehat{Z}(K; (\underline{X}, \underline{A}))$$

is defined to be the image of $Z(K; (\underline{X}, \underline{A}))$ in the smash product $X_1 \wedge X_2 \wedge \dots \wedge X_k$.

The image of $D(\sigma)$ in $\widehat{Z}(K; (\underline{X}, \underline{A}))$ is denoted by $\widehat{D}(\sigma)$ and is

$$Y_1 \wedge Y_2 \wedge \dots \wedge Y_k$$

where

$$Y_i = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \in [m] - \sigma. \end{cases}$$

In case it is important to distinguish the pair $(\underline{X}, \underline{A})$, the notations $D(\sigma; (\underline{X}, \underline{A}, *))$, and $\widehat{D}(\sigma; (\underline{X}, \underline{A}))$ will be used.

As in the case of $Z(K; (\underline{X}, \underline{A}))$, note that $\widehat{Z}(K; (\underline{X}, \underline{A}))$ is the colimit obtained from the spaces $\widehat{D}(\sigma)$.

Definition 2.5. Consider an ordered sequence $I = (i_1, \dots, i_k)$ with $1 \leq i_1 < \dots < i_k \leq m$ together with pointed spaces Y_1, \dots, Y_m . Then

- (1) the length of I is $|I| = k$,
- (2) the notation $I \subseteq [m]$ means I is any increasing subsequence of $(1, \dots, m)$,
- (3) $Y^{[m]} = Y_1 \times \dots \times Y_m$,
- (4) $Y^I = Y_{i_1} \times Y_{i_2} \times \dots \times Y_{i_k}$,
- (5) $\widehat{Y}^I = Y_{i_1} \wedge \dots \wedge Y_{i_k}$,

Given a sequence $I = (i_1, \dots, i_k)$ with $1 \leq i_1 < \dots < i_k \leq m$, define $K_I \subseteq K$ to be the *full sub-complex* of K consisting of all simplices of K which have all of their vertices in I , that is $K_I = \{\sigma \cap I \mid \sigma \in K\}$. This notation is used for the first decomposition proven in [1, 2] stated next.

Theorem 2.6. *Let K be an abstract simplicial complex with m vertices. Given $(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)\}_{i=1}^m$ where (X_i, A_i, x_i) are pointed triples of CW-complexes there is a natural pointed homotopy equivalence*

$$H : \Sigma(Z(K; (\underline{X}, \underline{A}))) \rightarrow \Sigma\left(\bigvee_{I \subseteq [m]} \widehat{Z}(K_I; (\underline{X}_I, \underline{A}_I))\right).$$

A second result in [1, 2] is stated next where $|lk_\sigma(K)|$ denotes the geometric realization of the link of σ in K .

Theorem 2.7. *Let K be an abstract simplicial complex with m vertices and \overline{K} its associated poset. Let $(\underline{X}, \underline{A})$ have the property that the inclusion $A_i \subset X_i$ is null-homotopic for all i . Then there is a homotopy equivalence*

$$\widehat{Z}(K; (\underline{X}, \underline{A})) \rightarrow \bigvee_{\sigma \in K} |\Delta(\overline{K}_{<\sigma})| * \widehat{D}(\sigma)$$

where

$$|\Delta(\overline{K}_{<\sigma})| = |lk_\sigma(K)|$$

the link of σ in K .

In particular if X_i is contractible for all i there is a homotopy equivalence

$$\widehat{Z}(K; (\underline{X}, \underline{A})) \rightarrow |K| * \widehat{A}^K.$$

Furthermore, there is a homotopy equivalence

$$\Sigma(Z(K; (\underline{X}, \underline{A}))) \rightarrow \Sigma\left(\bigvee_{I \in [m]} \left(\bigvee_{\sigma \in K_I} |\Delta((\overline{K}_I)_{<\sigma})| * \widehat{D}(\sigma)\right)\right).$$

Theorem 2.7 for the case X_i contractible for all i is called the *wedge lemma*.

A filtration on $Z(K; (\underline{X}, \underline{A}))$ is next described. The purpose of introducing this filtration is that there is an associated spectral sequence which is the subject of the article. The spectral sequence converges to the cohomology of $Z(K; (\underline{X}, \underline{A}))$.

Definition 2.8. *The $(m-1)$ -simplex $\Delta[m-1]$ is totally ordered by the left-lexicographical ordering of all faces defined as follows:*

$$\sigma = (i_1, i_2, \dots, i_s) < \tau = (j_1, j_2, \dots, j_t)$$

if and only if either

- (1) $1 \leq s < t \leq m$ or
- (2) $t = s$, and there exists an integer n such that $(i_1, i_2, \dots, i_n) = (j_1, j_2, \dots, j_n)$ but $i_{n+1} < j_{n+1}$.

There are

$$(1 + \binom{m}{1} + \binom{m}{2} + \binom{m}{3} + \dots + \binom{m}{m-1} + \binom{m}{m}) = 2^m$$

faces in $\Delta[m-1]$ (including the empty set) which are totally ordered by the integers q such that $0 \leq q \leq 2^m - 1$.

Furthermore, let σ_0 denote the empty set \emptyset ; thus $\sigma_0 \leq \sigma$ for all σ in $\Delta[m-1]$.

The weight of a face σ is that integer q , denoted by $wt(\sigma)$ where q is the position of σ in this total left lexicographical ordering of the simplices.

The $(m-1)$ -simplex $\Delta[m-1]$ is filtered by requiring

$$F_t \Delta[m-1] = \cup_{wt(\sigma) \leq t} \sigma.$$

This filtration of $\Delta[m-1]$ induces a filtration of K as given next.

Definition 2.9. The $(m-1)$ -simplex $\Delta[m-1]$ is filtered by the left lexicographical ordering of all faces as in Definition 2.8. Let K be a simplicial complex with m vertices.

Filter K by

$$F_t K = K \cap F_t \Delta[m-1].$$

Filter the polyhedral product $Z(K; X, A)$ and polyhedral smash product $\widehat{Z}(K; (X, A))$ by

(1)

$$F_t Z(K; (X, A)) = \cup_{wt(\sigma) \leq t} D(\sigma; (X, A)),$$

and

(2)

$$F_t \widehat{Z}(K; (X, A)) = \cup_{wt(\sigma) \leq t} \widehat{D}(\sigma; (X, A)).$$

Record this information stated as the next lemma.

Lemma 2.10. There is a total ordering of all of the faces of a simplicial complex K given by

- (1) the left-lexicographical ordering of all of the faces of $\Delta[m-1]$, and

(2) the induced ordering via the natural inclusion

$$K \subset \Delta[m-1].$$

Furthermore, inclusions

$$L \subset K$$

induced by an embedding of simplicial complexes with m vertices is order preserving, and filtration preserving where $F_t K = K \cap F_t \Delta[m-1]$ as listed in Definition 2.9. Namely, the inclusion $L \subset K$ induced by an embedding of simplicial complexes with m vertices is a morphism of filtered complexes

$$F_* L \subset F_* K.$$

This filtration of K induces a filtration of the polyhedral product $Z(K; X, A)$ and polyhedral smash product $\widehat{Z}(K; (X, A))$ given by

(1)

$$F_t Z(K; (X, A)) = \cup_{wt(\sigma) \leq t} D(\sigma; (X, A)),$$

and

(2)

$$F_t \widehat{Z}(K; (X, A)) = \cup_{wt(\sigma) \leq t} \widehat{D}(\sigma; (X, A)).$$

Furthermore, the natural quotient map $Z(K; X, A) \rightarrow \widehat{Z}(K; (X, A))$ is filtration preserving.

Remark 2.11. The filtration constructed in Lemma 2.10 is exploited in [4].

Definition 2.12.

(1) If $\sigma \in K$, write

$$(\underline{X}/\underline{A})^\sigma$$

for the smash product $(X_{i_1}/A_{i_1}) \wedge \cdots \wedge (X_{i_q}/A_{i_q})$ where $I = (i_1, \dots, i_q)$ is as in Definition 2.5 and σ has vertex set I .

(2) Write

$$\underline{A}^{\sigma^c}$$

for the product $A_{j_1} \times \cdots \times A_{j_{k-q}}$ where $\sigma \cup \{j_1, \dots, j_{k-q}\} = [m]$, and σ^c denotes the complement of σ . In particular for $\sigma = \emptyset$, $A^{\sigma^c} = A_1 \times \cdots \times A_k$.

Half-smash products are basic in this setting with their definition as follows.

Definition 2.13. *Let*

$$(X, x_0) \text{ and } (Y, y_0)$$

denote pointed spaces. Define

$$X \rtimes Y = (X \times Y)/(x_0 \times Y),$$

and

$$X \ltimes Y = (X \times Y)/(X \times y_0).$$

An example is given next.

Example 2.14. Let K denote the simplicial complex with two vertices $\{1, 2\}$ and with one edge $(1, 2)$. Then $Z(K; (\underline{X}, \underline{A})) = X_1 \times X_2$. The filtration of $X_1 \times X_2$ given in Definition 2.8 is stated next.

- (1) $F_0 Z(K; (\underline{X}, \underline{A})) = A_1 \times A_2$,
- (2) $F_1 Z(K; (\underline{X}, \underline{A})) = X_1 \times A_2$,
- (3) $F_2 Z(K; (\underline{X}, \underline{A})) = (X_1 \times A_2) \cup (A_1 \times X_2)$, and
- (4) $F_3 Z(K; (\underline{X}, \underline{A})) = X_1 \times X_2$.

Let

$$F_i = F_i Z(K; (\underline{X}, \underline{A}))$$

in this example. If (X_i, A_i) are pairs of finite CW-complexes, there are homeomorphisms

- (1) $F_1/F_0 \rightarrow (X_1/A_1 \times A_2)/(* \times A_2) \rightarrow X_1/A_1 \rtimes A_2$,
- (2) $F_2/F_1 = (X_1 \times A_2) \cup (A_1 \times X_2)/(X_1 \times A_2) \rightarrow A_1 \ltimes (X_2/A_2)$, and
- (3) $F_3/F_2 = X_1 \times X_2/(X_1 \times A_2 \cup A_1 \times X_2) \rightarrow (X_1/A_1) \wedge (X_2/A_2)$.

Letting $[x]$ denote image of the projection of $x \in X_1$ to X_1/A_1 then the homeomorphism in (2) is given by

$$(X_1 \times A_2) \cup (A_1 \times X_2)/(X_1 \times A_2) \simeq (A_1 \times X_2)/(A_1 \times A_2) \xrightarrow{p} A_1 \ltimes (X_2/A_2)$$

with $p(a \times x) = a \times [x]$

The homeomorphism in (3) is a special case of Lemma 3.2

The filtrations and their associated graded for the smash polyhedral products are exhibited next.

- (1) $F_0 \widehat{Z}(K; (\underline{X}, \underline{A})) = A_1 \wedge A_2$,
- (2) $F_1 \widehat{Z}(K; (\underline{X}, \underline{A})) = X_1 \wedge A_2$,
- (3) $F_2 \widehat{Z}(K; (\underline{X}, \underline{A})) = (X_1 \wedge A_2) \cup (A_1 \wedge X_2)$, and
- (4) $F_3 \widehat{Z}(K; (\underline{X}, \underline{A})) = X_1 \wedge X_2$.

Let

$$F_i = F_i \widehat{Z}(K; (\underline{X}, \underline{A}))$$

in this example where (X_i, A_i) are assumed to be pairs of finite CW-complexes. There are homeomorphisms

- (1) $F_1/F_0 \rightarrow (X_1/A_1) \wedge A_2$,
- (2) $F_2/F_1 = (X_1 \wedge A_2) \cup (A_1 \wedge X_2)/X_1 \wedge A_2 \rightarrow A_1 \wedge (X_2/A_2)$, and
- (3) $F_3/F_2 = X_1 \times X_2/(X_1 \times A_2 \cup A_1 \times X_2) \rightarrow (X_1/A_1) \wedge (X_2/A_2)$.

Given a filtered space, there is a natural spectral sequence associated to that filtration. The next theorem records the properties of the resulting spectral sequence of a filtered space in the context of polyhedral products with the left-lexicographical ordering obtained from Definition 2.8.

Theorem 2.15. *The left-lexicographical ordering of simplices induces spectral sequences of a filtered spaces*

- (1) $E_r(K; (\underline{X}, \underline{A})) \Rightarrow \widetilde{h}^*(Z(K; (\underline{X}, \underline{A})))$ with

$$E_1(K; (\underline{X}, \underline{A})) = \bigoplus_{\sigma \in K} \widetilde{h}^*((\underline{X}/\underline{A})^\sigma \rtimes \underline{A}^{\sigma^c}),$$

and a spectral sequence

- (2) $E_r(\widehat{K}; (\underline{X}, \underline{A})) \Rightarrow \widetilde{h}^*(\widehat{Z}(K; (\underline{X}, \underline{A})))$

$$E_1^{s,t}(\widehat{Z}(K; (\underline{X}, \underline{A}))) = \bigoplus_{\sigma \in K} \widetilde{h}^*((\underline{X}/\underline{A})^\sigma \wedge \widehat{\underline{A}}^{\sigma^c}).$$

The notation defined in 2.12. The grading, s is the index of the simplex, σ in the left-lexicographical ordering. t is the cohomological degree.

The differentials satisfy

$$d_r : E_r^{s,t} \rightarrow E_r^{s+r,t+1}.$$

Furthermore, the spectral sequence is natural for embeddings of simplicial maps, $L \subset K$ with the same number of vertices and with respect to maps of pointed pairs $(\underline{X}, \underline{A}) \rightarrow (\underline{Y}, \underline{B})$.

The natural quotient map

$$Z(K; (\underline{X}, \underline{A})) \rightarrow \widehat{Z}(K; (\underline{X}, \underline{A}))$$

induces a morphism of spectral sequences, and the stable decomposition of Theorem 2.6 induces a morphism of spectral sequences.

Remark 2.16. We remark that $Z(K; (\underline{X}, \underline{A}))$ and the spectral sequence commutes with colimits in $(\underline{X}, \underline{A})$.

We also note that we only use the fact that the inclusions $F_t \subset F_{t+1}$ are cofibrations. This follows from the hypothesis that (X_i, A_i) are finite CW pairs. The argument generalizes to NDR pairs.

Some consequences of this spectral sequence are worked out below. An explicit description of the cohomology of $Z(K; (\underline{X}, \underline{A}))$ with field coefficients \mathbb{F} will be given next followed by a section on examples. The answers for cohomology are given in terms of kernels and cokernels of

$$H^i(X_j) \rightarrow H^i(A_j).$$

Definition 2.17. Assume that K is a simplicial complex and the pointed pairs $(\underline{X}, \underline{A}, *)$ are of finite type. Assume that the maps $A_i \rightarrow X_i$ induce split surjections in cohomology with field coefficients \mathbb{F} . Consider the kernel of $H^i(X_j) \rightarrow H^i(A_j)$ together with the elements $x_j \in \ker(H^i(X_j) \rightarrow H^i(A_j))$ together with the two-sided ideal generated by all such $x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_t}$ with (i_1, i_2, \dots, i_t) not a simplex in K , denoted

$$SR(K; (\underline{X}, \underline{A})).$$

A result which is analogous to Theorem 2.35 of [1] follows next.

Theorem 2.18. Let K be an abstract simplicial complex with m vertices. Assume that

$$(\underline{X}, \underline{A}, *)$$

are pointed triples of connected CW-complexes of finite type for all i for which cohomology is taken with field coefficients \mathbb{F} . If the maps $A_i \rightarrow X_i$ induce split surjections in cohomology, then the induced map

$$H^*(X^{[m]}) \rightarrow H^*(Z(K; (\underline{X}, \underline{A})))$$

is an epimorphism of algebras which is additively split. Furthermore, there is an induced isomorphism of algebras

$$H^*(X^{[m]})/SR(K; (\underline{X}, \underline{A})) \rightarrow H^*(Z(K; (\underline{X}, \underline{A}))).$$

Recall that a map $A_i \rightarrow X_i$ induces a split monomorphism in integer homology if and only if it induces a split monomorphism with field coefficients for every prime field \mathbb{F}_p and the rational numbers. A corollary of Theorem 2.18 which follows immediately is stated next.

Corollary 2.19. *Let K be an abstract simplicial complex with m vertices. Assume that*

$$(\underline{X}, \underline{A}, *)$$

are pointed triples of connected CW-complexes of finite type for all i for which cohomology is taken with coefficients \mathbb{Z} . Assume that the maps $A_i \rightarrow X_i$ induce split monomorphisms in homology over \mathbb{Z} , then the induced map

$$H^*(X^{[m]}; \mathbb{Z}) \rightarrow H^*(Z(K; (\underline{X}, \underline{A})); \mathbb{Z})$$

is an epimorphism of algebras which is additively split.

Remark 2.20. The proof of Theorem 2.18 works just as well for any multiplicative cohomology h^* and CW pairs with $h^*(X_i)$ and $h^*(A_i)$ finitely generated free h^* modules.

3. A spectral sequence for the cohomology of $Z(K, (\underline{X}, \underline{A}))$ and $\widehat{Z}(K; (\underline{X}, \underline{A}))$

The object of this section is to construct the spectral sequences of Theorem 2.15. In subsequent sections these spectral sequences will be used to compute the cohomology of $Z(K; (\underline{X}, \underline{A}))$ when $(\underline{X}, \underline{A})$ satisfies suitable flatness conditions.

The spectral sequences of Theorem 2.15 are precisely those obtained by filtering the spaces $Z(K, (\underline{X}, \underline{A}))$ and $\widehat{Z}(K; (\underline{X}, \underline{A}))$ by finite filtrations induced by the left-lexicographical ordering. Since these spectral sequences arise by finite filtrations, the spectral sequences converge in the strong sense. It remains to identify the associated graded E_0 as well as E_1 , and the first differential.

The next three lemmas give the identification of E_0 .

Suppose that (X, x_0) and (Y, y_0) are both pointed CW complex, then the right and left half-smash products were defined in Definition 2.13 by

$$X \rtimes Y = (X \times Y)/(x_0 \times Y), \text{ and } X \ltimes Y = (X \times Y)/(X \times y_0).$$

A useful lemma follows in which X_+ denotes X with a disjoint base-point added.

Lemma 3.1. *Let*

$$(X, x_0), \text{ and } (Y, y_0)$$

be pointed, finite CW pairs. Then there are homotopy equivalences

- (1) $\Sigma(X \rtimes Y) \rightarrow \Sigma(X \wedge Y) \vee \Sigma(X)$,
- (2) $\Sigma(X \rtimes Y) \rightarrow \Sigma(X \wedge Y) \vee \Sigma(Y)$, and
- (3) $X \rtimes Y = (X \times Y)/(X \times y_0) \rightarrow X_+ \wedge Y$.

Lemma 3.2. *Let*

$$(X_i, A_i)$$

be finite CW pairs. Let Λ denote the subspace of

$$X_1 \times X_2 \times \cdots \times X_n$$

given by

$$\Lambda = \cup_{1 \leq i \leq m} X_1 \times X_2 \times \cdots \times X_{i-1} \times A_i \times X_{i+1} \times \cdots \times X_m.$$

There is a natural homeomorphism

$$\theta : X_1 \times X_2 \times \cdots \times X_m / \Lambda \rightarrow (X_1/A_1) \wedge (X_2/A_2) \wedge \cdots \wedge (X_m/A_m).$$

Proof. Let $[x_i]$ denote the image of $x_i \in X_i$ of the projection $X_i \rightarrow X_i/A_i$. The natural map

$$\phi : X_1 \times X_2 \times \cdots \times X_m \rightarrow (X_1/A_1) \wedge (X_2/A_2) \wedge \cdots \wedge (X_m/A_m)$$

which send (x_1, \dots, x_m) to $([x_1], \dots, [x_m])$ is a continuous surjection. The class (x_1, \dots, x_m) maps to the base point in $(X_1/A_1) \wedge (X_2/A_2) \wedge \cdots \wedge (X_m/A_m)$ if and only if at least one of the factors, x_i maps to the base point in X_i/A_i . Equivalently at least one of the factors $x_i \in A_i$. In particular ϕ factors through a map, θ . By construction θ is a bijection. The lemmas follow for compact CW complexes with A_i closed in X_i , since the target space is Hausdorff while the domain is compact. Thus the natural map is a homeomorphism.

□

Remark 3.3. The lemmas extends to locally finite CW complexes by taking a limit over finite skeleta.

Another useful lemma follows.

Lemma 3.4. *Let*

$$(Y_i, A_i)$$

be finite, pointed CW pairs. Then there is a homeomorphism

$$(Y_1 \times Y_2)/(A_1 \times Y_2) \rightarrow (Y_1/A_1) \rtimes (Y_2).$$

Thus there are homeomorphisms

$$(Y_1 \times Y_2 \times \cdots \times Y_n)/(A_1 \times Y_2 \times \cdots \times Y_n) \rightarrow (Y_1/A_1) \rtimes (Y_2 \times \cdots \times Y_n) \rightarrow (\cdots (Y_1/A_1) \rtimes (Y_2)) \rtimes Y_3 \cdots \rtimes Y_n).$$

Proof. The natural quotient map is a continuous bijection by inspection. Since all spaces are finite complexes with A_i closed in Y_i , the target space is Hausdorff while the domain is compact. Thus the natural map is a homeomorphism. \square

In the next lemma, abbreviate $F_s Z(K; (X, A))$ by $F_s Z$, and $F_s \widehat{Z}(K; (X, A))$ by $F_s \widehat{Z}$. Let $(X_i, A_i), i = 1, \cdots, m$ be finite, pointed CW pairs. The filtrations of Definition 2.9 are given by

$$(1) \quad F_t Z(K; (X, A)) = F_t Z = \cup_{wt(\sigma) \leq t} D(\sigma; (X, A)),$$

and

$$(2) \quad F_t \widehat{Z}(K; (X, A)) = F_t \widehat{Z} = \cup_{wt(\sigma) \leq t} \widehat{D}(\sigma; (X, A))$$

have the property that the natural quotient map

$$Z(K; X, A) \rightarrow \widehat{Z}(K; (X, A))$$

is filtration preserving, and are natural for morphisms of simplicial complexes

$$L \rightarrow K$$

which induce an isomorphism of sets on the vertices.

The next step is to identify the filtration quotients $F_s Z/F_{s-1} Z$ as well as $F_s \widehat{Z}/F_{s-1} \widehat{Z}$.

Lemma 3.5. *Let $(X_i, A_i), i = 1, \cdots, m$ be finite, pointed CW pairs. The filtrations of Definition 2.9 have the following property. If σ is the maximal simplex occurring in*

$$F_t Z(K; (X, A)) = F_t Z = \cup_{wt(\sigma) \leq t} D(\sigma; (X, A)),$$

then the natural quotient maps

$$F_t Z/F_{t-1} Z \rightarrow (\underline{X}/\underline{A})^\sigma \rtimes \underline{A}^{\sigma^c} \quad \text{and} \quad F_t \widehat{Z}/F_{t-1} \widehat{Z} \rightarrow (\underline{X}/\underline{A})^\sigma \wedge \widehat{\underline{A}^{\sigma^c}}$$

are homeomorphisms.

Proof. Suppose $F_s K$ is obtained from $F_{s-1} K$ by attaching an n -simplex Δ . For simplicity we may, after relabelling, assume Δ has vertices $\{1, 2, \dots, n+1\}$.

Δ and its boundary, $\partial\Delta$ can be viewed as simplicial complexes on the vertex set $[m]$. The vertices $\{n+2, \dots, m\}$ are not zero simplices. In the literature they are referred to as ghost vertices.

It follows from Grbić and Theriault [11] that there is a commutative diagram of cofibrations

$$(3.1) \quad \begin{array}{ccccc} Z(\partial\Delta; (\underline{X}, \underline{A})) & \xrightarrow{j} & Z(\Delta; (\underline{X}, \underline{A})) & \rightarrow & C \\ \downarrow & & \downarrow & & \parallel \\ F_{s-1}Z & \rightarrow & F_s Z & \rightarrow & C \end{array}$$

Now observe that

- (1) $Z(\Delta; (\underline{X}, \underline{A})) = X_1 \times \dots \times X_{n+1} \times A_{n+2} \times \dots \times A_m$.
- (2) $Z(\partial\Delta; (\underline{X}, \underline{A})) = Z(\partial\overline{\Delta}; (\underline{X}, \underline{A})) \times A_{n+2} \times \dots \times A_m$ where $\partial\overline{\Delta}$ is a simplicial complex on $[n+1]$. While $\partial\Delta$ is a simplicial complex on $[m]$ with the set of 0-simplices $= \{1, \dots, n+1\}$ and ghost vertices $= \{n+2, \dots, m\}$.
- (3) If $\partial\overline{\Delta} \subset [n+1]$ then

$$Z(\partial\overline{\Delta}; (\underline{X}, \underline{A})) = \bigcup_q X_1 \times \dots \times \mathcal{A}_q \times \dots \times X_{n+1}.$$

- (4) By lemma 3.2,

$$X_1 \times \dots \times X_{n+1} / Z(\partial\overline{\Delta}; (\underline{X}, \underline{A})) = (X_1/A_1) \wedge \dots \wedge (X_{n+1}/A_{n+1}).$$

- (5) It follows from Lemma 3.4 that for $A \subset X$ a CW pair the cofiber of $A \times Y \hookrightarrow X \times Y$ is the right half smash $(X/A) \rtimes Y$.

The following description of C follows from the top row of diagram 3.1 and these observations

$$C = F_s Z / F_{s-1} Z = (X_1/A_1) \wedge \dots \wedge (X_{n+1}/A_{n+1}) \rtimes (A_{n+2} \times \dots \times A_m).$$

A similar argument using part (2) of Lemma 3.2 shows that

Lemma 3.6.

$$\widehat{F}_s Z / \widehat{F}_{s-1} Z = (X_1/A_1) \wedge \dots \wedge (X_{n+1}/A_{n+1}) \wedge (A_{n+2} \wedge \dots \wedge A_m) = \widehat{D}(\sigma; (\underline{X/A}, \underline{A})).$$

This completes the description of the E_1 page and the proof of 2.15. □

Corollary 3.7. *Let $(X_i, A_i), i = 1, \dots, m$ be finite CW pairs. The filtrations of Definition 2.9 given by*

(1)

$$F_t Z(K; (X, A)) = F_t Z = \cup_{wt(\sigma) \leq t} D(\sigma; (X, A)),$$

and

(2)

$$F_t \widehat{Z}(K; (X, A)) = F_t \widehat{Z} = \cup_{wt(\sigma) \leq t} \widehat{D}(\sigma; (X, A))$$

have the property that the natural quotient map

$$Z(K; X, A) \rightarrow \widehat{Z}(K; (X, A))$$

is filtration preserving, and are natural for morphisms of simplicial complexes

$$L \rightarrow K$$

which induce an isomorphism of sets when restricted to the vertices.

Then there is a spectral sequence abutting to $h^*(Z(K; (X, A)))$ obtained from these filtrations for which the $E_0^{s,t}$ -term is specified by

The $E_1^{s,t}$ -term is specified by

$$(1) E_1^{s,t} Z(K; (X, A)) = h^t(F_s Z, F_{s-1} Z) = h^t(\underline{X}/\underline{A})^\sigma \rtimes \underline{A}^{\sigma^c}, \text{ and}$$

$$(2) E_1^{s,t} \widehat{Z}(K; (X, A)) = h^t(F_s \widehat{Z}, F_{s-1} \widehat{Z}) = h^t((\underline{X}/\underline{A})^\sigma \wedge \widehat{\underline{A}^{\sigma^c}}).$$

This spectral sequence has the following properties.

(1) The spectral sequence is natural for embeddings of simplicial complexes

$$L \subset K$$

with the same number of vertices.

(2) The spectral sequence is natural for morphisms of simplicial complexes

$$L \rightarrow K$$

which are order preserving (of the left lexicographical ordering).

(3) There is a finite filtration of $h^*(Z(K; (\underline{X}, \underline{A})))$ such that E_∞ is the associated graded group of this filtration.

An immediate application of the spectral sequence is a computation of $h^*(K; (\underline{X}, \underline{A}))$ as a ring when $h^*(X_i) \rightarrow h^*(A_i)$ is surjective for all i and a freeness condition is satisfied.

Proposition 3.8. *Suppose (X_i, A_i) is a CW pair, $h^*(X_i) \rightarrow h^*(A_i)$ is surjective for all i and $h^*(A_i)$ and $h^*(X_i/A_i)$ are free h^* modules. Then*

$$\tilde{h}^*(Z(K; (\underline{X}, \underline{A}))) = \bigoplus_{\sigma \in K} \tilde{h}^*((\underline{X}/\underline{A})^\sigma \rtimes \underline{A}^{\sigma^c}).$$

Proof. The hypothesis implies $\tilde{h}^*(X_i/A_i) \rightarrow h^*(X_i)$ is injective, and by the freeness assumption, $h^*(C) \rightarrow h^*(Z(\Delta; (\underline{X}, \underline{A})))$ in the following diagram is injective.

$$\begin{array}{ccccccc} 0 \leftarrow & h^*(Z(\partial\Delta; (\underline{X}, \underline{A}))) & \leftarrow & h^*(Z(\Delta; (\underline{X}, \underline{A}))) & \leftarrow & h^*(C) & \leftarrow 0 \\ & \uparrow & & \uparrow & & \parallel & \\ & \leftarrow h^*(Z(K_{q-1}, (\underline{X}, \underline{A}))) & \leftarrow & h^*(Z(K_q, (\underline{X}, \underline{A}))) & \xleftarrow{j} & h^*(C) & \xleftarrow{\delta} \end{array}$$

which implies j is injective. Hence $\delta = 0$. This implies the differentials in the spectral sequence are zero. □

This is particularly interesting in the cases where the surjectivity or the freeness conditions are satisfied for h^* , but not for ordinary cohomology. For example $h^* = K^*$ and $(X_i, A_i) = (SO(2n+1), SO(2n))$.

Specializing h^* to ordinary cohomology with coefficients in a field we can now prove theorem 2.18. The surjectivity condition of Proposition 3.8 implies $H^*(X_i/A_i)$ is a subring of $H^*(X_i)$. Therefore I in the following corollary is an ideal in $H^*(X_1) \otimes \cdots \otimes H^*(X_m)$.

Corollary 3.9. *Assume H^* is cohomology with coefficients in a field and that (X_i, A_i) is a CW pair, such that $H^*(X_i) \rightarrow H^*(A_i)$ is surjective for all i . Then there is a ring isomorphism*

$$H^*(Z(K; (\underline{X}, \underline{A}))) = H^*(X_1) \otimes \cdots \otimes H^*(X_m)/I$$

where I is the ideal generated by $\tilde{H}^*(X_{j_1}/A_{j_1}) \otimes \cdots \otimes \tilde{H}^*(X_{j_t}/A_{j_t})$, with (j_1, \dots, j_t) not spanning a simplex in K .

Proof. The hypothesis implies there is a split short exact sequence

$$0 \rightarrow \tilde{H}^*(X_i/A_i) \rightarrow H^*(X_i) \rightarrow H^*(A_i) \rightarrow 0$$

for all $i > 0$.

After choosing a splitting we may write $H^*(X_i) = H^*(A_i) \oplus \tilde{H}^*(X_i/A_i)$.

The tensor product $H^*(X_1) \otimes \cdots \otimes H^*(X_m)$ may now be written as a sum of terms of the form $H^*(Y_1) \otimes \cdots \otimes H^*(Y_m)$ where Y_i is A_i or X_i/A_i . In particular Proposition 3.8 implies

the natural map of rings $H^*(X_1) \otimes \cdots \otimes H^*(X_m) \rightarrow H^*(Z(K; (\underline{X}, \underline{A})))$ is surjective with kernel given by the ideal I . \square

The Stanley-Reisner ring is the cohomology ring in the special case, $(\underline{X}, \underline{A}) = (CP^\infty, *)$

Example 3.10. It follows from [7] and [5, Example 6.40] that $Z(K; (D^1, S^0))$ is an orientable surface if K is the boundary of an n -gon. The next application of the spectral sequence is to determine the genus of this surface. The genus is determined by the Euler characteristic of $H^*(Z(K; (D^1, S^0)))$ which is the Euler characteristic of the E^1 page.

The boundary of an n -gon has n 0-simplices and n 1-simplices. To compute the Euler characteristic, the ranks of $\tilde{H}^*((\underline{X}/\underline{A})^\sigma \rtimes \underline{A}^{\sigma^c})$ are computed.

- (1) For $\sigma = \emptyset$, $H^*((\underline{X}/\underline{A})^\sigma \rtimes \underline{A}^{\sigma^c}) = H^*(\overbrace{S^0 \times \cdots \times S^0}^n)$ where $\overbrace{S^0 \times \cdots \times S^0}^n$ is 2^n distinct points.

So the unreduced homology has 2^n 0-dimensional classes.

- (2) For σ a 0-simplex

$$\begin{aligned} (\underline{X}/\underline{A})^\sigma \rtimes \underline{A}^{\sigma^c} &= \\ S^1 \rtimes \overbrace{S^0 \times \cdots \times S^0}^{n-1} &= \\ S^1 \wedge (2^{n-1} \text{ points})_+ &= S^1 \wedge \left(\bigvee_{2^{n-1}} S^0 \right) \end{aligned}$$

So $\tilde{H}^*((\underline{X}/\underline{A})^\sigma \rtimes \underline{A}^{\sigma^c})$ has 2^{n-1} 1-dimensional classes. There are n 0-simplices so there are a total of $n(2^{n-1})$ 1-dimensional classes.

- (3) If σ is a 1-simplex, the computation is similar.

$$\begin{aligned} (\underline{X}/\underline{A})^\sigma \rtimes \underline{A}^{\sigma^c} &= \\ S^2 \rtimes \overbrace{S^0 \times \cdots \times S^0}^{n-2} &= \\ S^2 \wedge (2^{n-2} \text{ points})_+ &= S^2 \wedge \left(\bigvee_{2^{n-2}} S^0 \right) \end{aligned}$$

which contributes 2^{n-2} 2-dimensional classes. There are n 1-simplices. So there are a total of $n(2^{n-2})$ 2-cells.

So the Euler characteristic of E_1 and hence of $Z(K; (D^1, S^0))$ is $(4 - n)2^{n-2}$. This proves a theorem of Coxeter, [7], i.e. if K is the boundary of an n -gon $Z(K; (D^1, S^0))$ is a surface of genus $1 + (n - 4)2^{n-3}$.

M. Davis first computed the Euler characteristic of $Z(K; (D^1, S^0))$ [8], but the analogous spectral sequence argument as given in the above example also gives the following result.

Proposition 3.11. *if K is a simplicial complex with m vertices which has t_n n -simplices then*

$$\chi(Z(K; (D^1, S^0))) = \Sigma(-1)^{n+1} t_n 2^{m-n-1}.$$

(The empty simplex is considered to be a (-1) -simplex, $t_{-1} = 1$).

4. Computing the differentials.

For CW pairs satisfying a freeness condition, the differentials in the spectral sequence are shown to be determined by the coboundaries of the long exact sequences of the pairs (X_i, A_i) . This result is used to compute the generalized cohomology of $\widehat{Z}(K; (\underline{X}, \underline{A}))$.

We recall the description of the E_1 page of the spectral sequences constructed in section 3.

(4.1)

(1) $E_r(Z(K; (\underline{X}, \underline{A}))) \Rightarrow \widetilde{h}^*(Z(K; (\underline{X}, \underline{A})))$ with

$$E_1(Z(K; (\underline{X}, \underline{A}))) = \bigoplus_{\sigma \in K} \widetilde{h}^*((\underline{X}/\underline{A})^\sigma \rtimes (A^{\sigma^c})).$$

(2) $E_r(\widehat{Z}(K; (\underline{X}, \underline{A}))) \Rightarrow \widetilde{h}^*(\widehat{Z}(K; (\underline{X}, \underline{A})))$ with

$$E_1(\widehat{Z}(K; (\underline{X}, \underline{A}))) = \bigoplus_{\sigma \in K} \widetilde{h}^*((\underline{X}/\underline{A})^\sigma \wedge (\widehat{A}^{\sigma^c})).$$

We next compute the differentials in the spectral sequence. In order to do so the strong freeness assumption, Definition 2.2, on the pairs (X_i, A_i) is imposed.

Using the Künneth theorem the E_1 page of the spectral spectral sequence converging to $\widetilde{h}^*(\widehat{Z}(K; (\underline{X}, \underline{A})))$ is isomorphic to a direct sum of h^* -modules

$$\tilde{h}^*((\underline{X}/\underline{A}))^\sigma \bigotimes_{j_i \notin \sigma} \otimes (\tilde{h}^*(A_{j_i}))$$

Each summand is a tensor product of the cohomology of X_i/A_i (which is a sum of C_i and W_i) and the cohomology of A_i (which is the sum of E_i and B_i). After expanding the tensor product the E_1 page becomes a sum of tensor products of E_i, C_i, B_i and W_i . Hence any class in $E_1^{s,t}$ is a sum of classes

$$y_1 \otimes \cdots \otimes y_m$$

with $y_j \in E_i, C_i, B_i$ or W_i .

There are coboundary maps

$$\delta_i : H^*(A_i) \supset E_i \rightarrow W_i \subset H^{*+1}(X_i/A_i).$$

Definition 4.1. *There is a coboundary map, δ defined on the h^* -generators $y_1 \otimes \cdots \otimes y_m$ of*

$$\tilde{h}^*((\underline{X}/\underline{A}))^\sigma \bigotimes_{j_i \notin \sigma} \otimes (\tilde{h}^*(A_{j_i}))$$

by the coboundary maps δ_i and the graded Leibniz rule.

A monomial,

$$y_1 \otimes \cdots \otimes y_m \in \tilde{h}^*((\underline{X}/\underline{A}))^\sigma \bigotimes_{j_i \notin \sigma} \otimes (\tilde{h}^*(A_{j_i}))$$

defines a simplex, $\sigma(y_1 \otimes \cdots \otimes y_m)$ as follows. There are two indexing sets determined by $y_1 \otimes \cdots \otimes y_m$.

$$\begin{aligned} I_1 &= \{i \in [m] \mid y_i \in C_i\} \\ I_2 &= \{i \in [m] \mid y_i \in W_i\} \end{aligned}$$

The h^* -modules C_i and W_i are summands of $\tilde{h}^*(X_i/A_i)$. The E_1 page of the spectral sequence is a sum of terms with factors of $\tilde{h}^*(X_i/A_i)$ indexed by the simplices of K . It follows that $\sigma(y_1 \otimes \cdots \otimes y_m) = I_1 \cup I_2$ is a simplex of K .

The weight of $\sigma(y_1 \otimes \cdots \otimes y_m)$ in the left-lexicographical ordering of the simplices of K is the filtration of $y_1 \otimes \cdots \otimes y_m$ in the spectral sequence. The simplex $\sigma(y_1 \otimes \cdots \otimes y_m)$ is called the support of $y_1 \otimes \cdots \otimes y_m$.

Theorem 4.2. *Assume $(\underline{X}, \underline{A})$ satisfies the strong freeness condition in definition 2.2, then*

$$y = \sum_{\ell} y_{1_\ell} \cdots y_{m_\ell}$$

survives to $E_r^{s,*}$ if

- (1) $\sigma(y_{1_\ell} \cdots y_{m_\ell})$ has weight s for all ℓ

and, with δ as in Definition 4.1

- (2)

$$\delta(y) = \sum_t \bar{y}_{1_t} \cdots \bar{y}_{m_t}$$

where

$$\text{weight } \sigma(\bar{y}_{1_t} \cdots \bar{y}_{m_t}) \geq s + r$$

for all t .

Then

$$d_r(y) = \sum_{wt(\sigma(\bar{y}_{1_t} \cdots \bar{y}_{m_t})) = s+r} \bar{y}_{1_t} \cdots \bar{y}_{m_t}$$

Example 4.3. We illustrate Theorem 4.2 for $h^* = H^*$, K a simplicial complex on 3 points, i.e. $K = 3$ distinct points, $K =$ an edge and a disjoint point, $K =$ two edges meeting at a common vertex, $K =$ the boundary of the 2 simplex and finally $K =$ the 2 simplex. $(\underline{X}, \underline{A}) = (D^1, S^0)$. There are the generators, $e_0 \in \tilde{H}^0(S^0)$ and $w_1 \in \tilde{H}^1(S^1)$.

We first build up the spectral sequence for three distinct points then add one edge at a time followed by the two simplex.

- Three distinct points.

filtration	0	1	2	3
	$e_0 \otimes e_0 \otimes e_0$	$w_1 \otimes e_0 \otimes e_0$	$e_0 \otimes w_1 \otimes e_0$	$e_0 \otimes e_0 \otimes w_1$

4.2 implies there is a differential from filtration 0 to filtration 1.

- Add one edge.

filtration	0	1	2	3	4
	0	0	$e_0 \otimes w_1 \otimes e_0$	$e_0 \otimes e_0 \otimes w_1$	$w_1 \otimes w_1 \otimes e_0$

4.2 implies a differential from filtration 2 to filtration 4.

- Add another edge

filtration	0	1	2	3	4	5
	0	0	0	$e_0 \otimes e_0 \otimes w_1$	0	$w_1 \otimes e_0 \otimes w_1$

4.2 implies a differential from filtration 3 to 5.

- Add another edge to form $\partial\Delta^2$

$$\begin{array}{cccccccc} \text{filtration} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ & 0 & 0 & 0 & 0 & 0 & 0 & e_0 \otimes w_1 \otimes w_1 \end{array}$$

- Add Δ^2

$$\begin{array}{ccccccccccc} \text{filtration} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & & 7 \\ & 0 & 0 & 0 & 0 & 0 & 0 & e_0 \otimes w_1 \otimes w_1 & w_1 \otimes w_1 \otimes w_1 \end{array}$$

There is a differential from filtration 6 to filtration 7.

Proof of Theorem 4.2. The filtration in the spectral sequence is induced by the left-lexicographical order of the simplices which are added one at a time. The weight of a simplex is its position in this order. In particular the empty simplex has index 0. The proof is an induction on the index of the simplex being added. In the proof, $\tilde{h}^*(F_s \hat{K})$ is written F_s . Inductively assume F_{s-1} is given by Theorem 4.2. The induction starts with $F_0 = \tilde{h}^*(A_1) \otimes \cdots \otimes \tilde{h}^*(A_m)$.

Recall the definition of the differentials in the spectral sequence. The spectral sequence is induced by the exact couple

$$\begin{array}{ccc} F_{t-1} & \xleftarrow{\tilde{h}^*(i)} & F_t \\ & \searrow \delta & \nearrow \tilde{h}^*(j) \\ & \tilde{h}^*(\hat{D}(\sigma_t; (\underline{X}/\underline{A}, \underline{A}))) & \end{array}$$

Where the maps are induced by the inclusions, $i : F_{t-1} \hat{K} \rightarrow F_t \hat{K}$, and the projections $j : F_t \hat{K} \rightarrow F_t \hat{K} / F_{t-1} \hat{K} = \hat{D}(\sigma_t; (\underline{X}/\underline{A}, \underline{A}))$ where $\hat{D}(\sigma_t; (\underline{X}/\underline{A}, \underline{A}))$ is defined in Lemma 3.6. δ denotes the coboundary map.

A commutative diagram with four vertices. The top-left vertex is labeled F_t . The top-right vertex is labeled F_{s-1} and has an additional label $\beta \in$ next to it. The bottom-left vertex is labeled $\alpha \in \tilde{h}^*(\widehat{D}(\sigma_t; (\underline{X}/\underline{A}, \underline{A})))$. The bottom-right vertex is labeled $\gamma \in h^{*+1}(\widehat{D}(\sigma_s; (\underline{X}/\underline{A}, \underline{A})))$. An arrow points from F_t to the bottom-left vertex, labeled $\tilde{h}^*(j)$. An arrow points from F_{s-1} to the bottom-right vertex, labeled δ . A horizontal arrow points from F_{s-1} to F_t , labeled $\tilde{h}^*(i)$.

Commutative diagram showing relationships between various spaces and maps:

- Top row: $\tilde{h}^*(\widehat{Z}(\partial\Delta; (\underline{X}, \underline{A})))$ and $\tilde{h}^*(\widehat{Z}(\Delta; (\underline{X}, \underline{A})))$
- Second row: F_{t-1} , F_t , F_{s-1} , F_s
- Bottom row: $\alpha \in \tilde{h}^*(\widehat{D}(\sigma_t; (\underline{X}/\underline{A}, \underline{A})))$ and $h^{*+1}(\widehat{D}(\sigma_s; (\underline{X}/\underline{A}, \underline{A})))$

Maps and connections:

- $\tilde{h}^*(i): F_{t-1} \leftarrow F_t$ (dashed arrow)
- $\tilde{h}^*(i): F_{s-1} \leftarrow F_s$ (dashed arrow)
- $\tilde{h}^*(j): \alpha \rightarrow F_t$ (solid arrow)
- $\delta: F_{s-1} \rightarrow h^{*+1}(\widehat{D}(\sigma_s; (\underline{X}/\underline{A}, \underline{A})))$ (solid arrow)
- $\tilde{\delta}: F_t \rightarrow h^{*+1}(\widehat{D}(\sigma_s; (\underline{X}/\underline{A}, \underline{A})))$ (solid arrow)
- $\theta: F_{s-1} \rightarrow \tilde{h}^*(\widehat{Z}(\partial\Delta; (\underline{X}, \underline{A})))$ (solid arrow)
- $\tilde{h}^*(\widehat{Z}(\Delta; (\underline{X}, \underline{A}))) \rightarrow \tilde{h}^*(\widehat{Z}(\partial\Delta; (\underline{X}, \underline{A})))$ (solid arrow)
- $F_t \rightarrow F_{s-1}$ (solid arrow)

$$\hat{Z}(\partial\Delta; (\underline{X}, \underline{A})) \rightarrow \hat{Z}(\Delta; (\underline{X}, \underline{A})) \rightarrow \hat{D}(\sigma_s; (\underline{X}/\underline{A}, \underline{A})).$$

By induction β is a sum of terms of the form $y_1 \otimes \cdots \otimes y_m$. We write $(y_1 \otimes \cdots \otimes y_m)_\ell$ for the terms that appear in β

$$\beta = \sum_{\ell} (y_1 \otimes \cdots \otimes y_m)_\ell.$$

The differential of α is given by

$$d(\alpha) = \tilde{\delta}(\theta(\sum_{\ell} (y_1 \otimes \cdots \otimes y_m)_\ell))$$

where $\tilde{\delta}$ is given by the formula in part (3) of Theorem 4.2.

□

The following Lemma ??, motivates the definition of *strongly isomorphic h^* -homology*, which was also defined in [4].

Definition 4.4. *The pairs $(\underline{Y}, \underline{B})$ and $(\underline{X}, \underline{A})$ are said to have **strongly isomorphic h^* -cohomology** provided*

(1) *there are h^* -isomorphisms*

$$\beta_j : h^*(Y_j) \rightarrow h^*(X_j),$$

$$\alpha_j : h^*(B_j) \rightarrow h^*(A_j),$$

and

$$\gamma_j : h^*(Y_j/B_j) \rightarrow h^*(X_j/A_j)$$

(2) *there is an commutative diagram of exact sequences*

$$\begin{array}{ccccccc} \longrightarrow & \bar{H}_i(B_j) & \xrightarrow{\lambda_{j*}} & \bar{H}_i(Y_j) & \longrightarrow & H_*(Y_j/B_j) & \xrightarrow{\delta} \longrightarrow \\ & \downarrow \alpha_j & & \downarrow \beta_j & & \downarrow \gamma_j & \\ \longrightarrow & \bar{H}_i(A_j) & \xrightarrow{\iota_{j*}} & \bar{H}_i(X_j) & \longrightarrow & H_*(X_j/A_j) & \xrightarrow{\delta} \longrightarrow \end{array}.$$

where $\lambda_j : B_j \subset Y_j$, and $\iota_j : A_j \subset X_j$ are the natural inclusions.

(3) *The maps of triples*

$$(\alpha_j, \beta_j, \gamma_j) : (H_*(Y_j), H_*(B_j), H_*(Y_j/B_j)) \rightarrow (H_*(X_j), H_*(A_j), H_*(X_j/A_j))$$

which satisfy conditions 1–2 are said to **induce a strong homology isomorphism**.

We will sometimes say that pairs of h^* modules with maps satisfying conditions 1 – 3 are strongly h^* isomorphic without reference to any spaces. The next corollary is an immediate consequence of 4.2.

Corollary 4.5. *Assume (X_i, A_i) are CW pairs for all i which satisfy the freeness conditions of 4.2. Then the spectral sequence $E_r(Z(K; (\underline{X}, \underline{A}))) \Rightarrow \tilde{h}^*(Z(K; (\underline{X}, \underline{A})))$ depends only on the strong h^* cohomology isomorphism type of the pairs, (X_i, A_i) .*

Specifically the filtration, differentials and extensions depend only on K and the h^* cohomology isomorphism type of the pairs, (X_i, A_i) .

A straightforward, recursive application of the splitting for the right smash, Lemma 3.1, shows that there is a homotopy equivalence

$$(4.2) \quad \Sigma(\bigvee_{I \subset [m]} [\bigvee_{\sigma \in K_I} (\underline{X}/\underline{A})^\sigma \wedge (\hat{A}^{I-|\sigma|})]) \rightarrow \Sigma(\bigvee_{\sigma \in K} (\underline{X}/\underline{A})^\sigma \rtimes (A^{\sigma^c})).$$

This implies a splitting of $E_1(Z(K; (\underline{X}, \underline{A})))$ into a sum over I of $E_1(\hat{Z}(K_I; (\underline{X}, \underline{A})))$. Hence Theorem 2.6 appears at the level of the E_1 pages of the spectral sequences in equation (4.1). The differentials respect the splitting. The next corollary records this.

Corollary 4.6. *Assume (X_i, A_i) are CW pairs which satisfy the freeness conditions of 4.2. Then*

$$E_r(Z(K; (\underline{X}, \underline{A}))) = \bigoplus_{I \subset [m]} E_r(\hat{Z}(K_I; (\underline{X}, \underline{A})_I)).$$

The following example illustrates the main ideas in the proof. The notation is as in (2.2).

(4.3)

- The splitting at E_1 is a consequence of the presence of the classes $1 \in B_i$ defined in 2.2 which appears in $E_r(Z(K; (\underline{X}, \underline{A})))$ but not in $E_r(\hat{Z}(K_I; (\underline{X}, \underline{A})_I))$.
- The differentials only involve E_i and W_i , hence the spectral sequences are modules over B_i and C_i .

Example 4.7. Let h^* = ordinary cohomology, K the simplicial complex of two disjoint points, $\{1, 2\}$, and $X_i = D^2 \vee S^3$, $A_i = S^1 \subset D^2 \subset X_i$ for $i = 1, 2$. Since i is determined by the coordinate in the subsequent tensor products, we omit it from the notation.

The modules of (2.2) are as follows:

$$\begin{aligned} E &= \mathbb{Z}\{e_1\} \in H^1(S^1), \quad B = \mathbb{Z}\{1\} \in H^0(D^2 \bigvee S^3), \\ C &= \mathbb{Z}\{c_3\} \in H^3(D^2 \bigvee S^3), \quad W = \mathbb{Z}\{w_2\} \in H^2((D^2/S^1) \bigvee S^3). \end{aligned}$$

For this case, the decomposition given by equation (4.1) has the form

$$\begin{aligned} E_1(Z(K; (\underline{X}, \underline{A}))) &= \begin{array}{ccccc} H^*(A \times A) & \oplus & H^*(X/A \rtimes A) & \oplus & H^*(A \ltimes X/A) \\ \parallel & & \parallel & & \parallel \\ (e_1 \oplus 1) \otimes (1 \oplus e_1) & \oplus & (c_3 \oplus w_2) \otimes (e_1 \oplus 1) & \oplus & (e_1 \oplus 1) \otimes (c_3 \oplus w_2) \end{array} \end{aligned}$$

which equals

$$\begin{aligned} &(1 \otimes 1) \oplus (e_1 \otimes 1) \oplus (1 \otimes e_1) \oplus (e_1 \otimes e_1) \\ &\quad \oplus \\ &(c_3 \otimes e_1) \oplus (c_3 \otimes 1) \oplus (w_2 \otimes e_1) \oplus (w_2 \otimes 1) \\ &\quad \oplus \\ &(e_1 \otimes c_3) \oplus (1 \otimes c_3) \oplus (e_1 \otimes w_2) \oplus (1 \otimes w_2) \end{aligned}$$

Now arrange the summands according to the location of the unit

$$\begin{aligned} &(e_1 \otimes e_1) \oplus (c_3 \otimes e_1) \oplus (w_2 \otimes e_1) \oplus (e_1 \otimes c_3) \oplus (e_1 \otimes w_2) \\ &(e_1 \otimes 1) \oplus (c_3 \otimes 1) \oplus (w_2 \otimes 1) \\ &(1 \otimes e_1) \oplus (1 \otimes c_3) \oplus (1 \otimes w_2) \\ &(1 \otimes 1) \end{aligned}$$

The first line is $E_1(\widehat{Z}(K_{\{1,2\}}; (\underline{X}, \underline{A})))$, the second line is $E_1(\widehat{Z}(K_{\{1\}}; (\underline{X}, \underline{A})))$ and the third line is $E_1(\widehat{Z}(K_{\{2\}}; (\underline{X}, \underline{A})))$. The last line is the unit in $H^0(Z(K; (\underline{X}, \underline{A})))$ which appears as $E_1(\widehat{Z}(K_{\emptyset}; (\underline{X}, \underline{A})))$. This decomposition is an example of the first observation of (4.3).

There is a differential in the first line from $e_1 \otimes e_1$ to $w_2 \otimes e_1$, a differential in the second line from $e_1 \otimes 1$ to $w_2 \otimes 1$ and a similar differential from $1 \otimes e_1$ to $1 \otimes w_2$. The differentials are zero on 1 and c_3 . This illustrates the second point of 4.3.

Using the units we have splitting of the spectral sequence.

$$(4.4) \quad E_r(Z(K; (\underline{X}, \underline{A}))) = \bigoplus_{I \subset [m]} E_r(\widehat{Z}(K_I; (\underline{X}, \underline{A})_I)) \otimes 1^{[m]-I}$$

where $1^{[m]-I}$ denotes a factor of 1 in each coordinate in the complement of I .

Proof of Corollary 4.6. First some equalities that were used in 4.3 and are relevant to proving 4.6 that follow from the Künneth formula are listed. We assume X and Y have free h^* cohomology.

$$(1) \quad \tilde{h}^*(X \times Y) = (\tilde{h}^*(X) \otimes \tilde{h}^*(Y)) \oplus (\tilde{h}^*(X) \otimes 1) \oplus (1 \otimes \tilde{h}^*(Y)).$$

This is the algebraic version of the homotopy splitting

$$\Sigma(X \times Y) = \Sigma(X \wedge Y) \vee \Sigma(X) \vee \Sigma(Y).$$

$$(2) \quad \tilde{h}^*(X \rtimes Y) = [\tilde{h}^*(X) \otimes \tilde{h}^*(Y)] \oplus [\tilde{h}^*(X) \otimes 1].$$

This is the algebraic version of Lemma 3.1.

(3) More generally, with $\sigma = \{1, 2, \dots, n\}$.

$$\begin{aligned} & h^*((\underline{X}/\underline{A})^\sigma \rtimes \underline{A}^{\sigma^c}) \\ &= \tilde{h}^*((X_1/A_1) \wedge \dots \wedge (X_n/A_n) \bigwedge (A_{n+1} \times \dots \times A_m)) \bigoplus \tilde{h}^*((X_1/A_1) \wedge \dots \wedge (X_n/A_n)) \otimes 1 \dots \otimes 1 \\ &= \bigoplus_{\{i_1, \dots, i_p\} \subset \{1, \dots, n\}} \tilde{h}^*((X_1/A_1)) \otimes \dots \otimes \tilde{h}^*(X_n/A_n) \otimes \tilde{h}^*(A_{i_1}) \otimes \dots \otimes \tilde{h}^*(A_{i_p}) \otimes 1 \otimes \dots \otimes 1 \\ & \quad \text{where the units appear in the factors with coordinates in } [m] \text{ not in the set } \{i_1, \dots, i_p\}. \end{aligned}$$

Summing over all simplices gives the isomorphism in Corollary 4.6. \square

5. The Cohomology of the polyhedral product.

The freeness conditions of Section 4 are assumed for $(\underline{X}, \underline{A})$ throughout this section the goal of which is to compute $h^*(\widehat{Z}(K; (\underline{X}, \underline{A})))$ in terms of the strong h^* -cohomology isomorphism type of the pairs (X_i, A_i) and the cohomology of sub-complexes of K . A consequence is a formula for the reduced Poincare series for $\tilde{H}^*(Z(K; (\underline{X}, \underline{A})))$.

K denotes a simplicial complex with m vertices. E_i, B_i, C_i and W_i are as in Definition 2.2.

Write E_1 for $E_1(\widehat{Z}(K; (\underline{X}, \underline{A})))$.

Recall the following from Section 4.

- (1) E_1 is a sum of h^* -modules

$$\tilde{h}^*((\underline{X}/\underline{A})^\sigma \wedge (\hat{A}^{\sigma^c}))$$

which is a sum of tensor products of E_i, B_i, C_i, W_i .

In particular a typical summand in E_1 may be written in the form

$$(5.1) \quad E^J \otimes W^L \otimes C^S \otimes B^T,$$

with $J \cup L \cup S \cup T = [m]$ and J, L, S, T mutually disjoint.

- (2) The indexing set $L \cup S$ is a simplex in K (see Definition 4.1).
- (3) By Theorem 4.2 the differentials in the spectral sequence are induced by the coboundary maps, $\delta_i : \tilde{h}^*(A_i) \rightarrow \tilde{h}^*(X_i/A_i)$. with δ_i mapping E_i to W_i .

We next show that E_r is a sum of simpler spectral sequences. To this end fix S and T . The next definition, which is simply a reparametrization of $C^S \otimes B^T$ is for convenient bookkeeping.

Definition 5.1. Let $I \subset [m]$, $I = (i_1, \dots, i_p)$.

Let $\sigma \subset I$ be a simplex in K . Define

$Y^{I,\sigma} = Y_1 \otimes \dots \otimes Y_p$ where

$$Y_t = \begin{cases} C_{i_t} & \text{if } i_t \in \sigma \\ B_{i_t} & \text{if } i_t \notin \sigma \end{cases}$$

In the notation of (5.1) $\sigma = S$ and $I = S \cup T$.

By definition, $Y^{I,\sigma}$ is fixed because S and T are fixed. Now consider the sum

$$\left[\bigoplus_{\substack{J \cup L \cup I = [m] \\ L \cup \sigma \in K}} E^J \otimes W^L \right] \otimes Y^{I,\sigma}$$

which is a sub-sum of the E_1 page of the spectral sequence. Since J, L and I are mutually disjoint, and $L \cup \sigma \in K$ it follows that L is a simplex of K belonging to the link of σ in the complement of I , which is now defined.

Definition 5.2. If $I \subset [m]$ and $\sigma \in K$, $\sigma \subset I$ then the link of σ in the complement of I , $N(I, \sigma)$, is the simplicial complex, $N(I, \sigma)$ on vertex set $[m] - I$ such that

$$\bar{\sigma} \in N(I, \sigma) \Leftrightarrow \bar{\sigma} \cup \sigma \in K.$$

Note that

- (1) $N(I, \sigma)$ is indeed a simplicial complex since $\sigma' \subset \bar{\sigma}$ implies $\sigma' \cup \sigma \in K$ which implies $\sigma' \in N(I, \sigma)$.
- (2) If $N(I, \sigma) = \emptyset$ then $\sigma \cup \{v\} \notin K$ for all $v \in [m] - I$. This implies $N(I, \sigma) = \emptyset \Leftrightarrow \sigma$ is a maximal simplex in K .
- (3) $N(I, \emptyset) = K_{[m]-I}$.
- (4) $N(|\sigma|, \sigma) = lk_\sigma(K)$, the link of σ in K .

Since all differentials take place between the terms E^J and W^L in E_r it follows that for fixed I, σ

$$\left[\bigoplus_{\substack{J \cup L \cup I = [m] \\ L \cup \sigma \in K}} E^J \otimes W^L \right] \otimes Y^{I, \sigma}$$

is a sub spectral sequence of E_r with all differentials taking place within the brackets. In particular

$$\left[\bigoplus_{\substack{J \cup L \cup I = [m] \\ L \cup \sigma \in K}} E^J \otimes W^L \right]$$

is the E_1 -page of a spectral sequence.

This spectral sequence is next identified as a spectral sequence

$$\left[E_r(\widehat{Z}(N(I, \sigma); (\underline{CV}, \underline{V}))) \right]$$

for some collection of CW complexes, $\{V_i\}$.

Recall that $h^*(A_i) = E_i \oplus B_i$ for free h^* -modules E_i, B_i . Let

$$\{e_i^\ell\}$$

be a set of generators for E_i .

Definition 5.3. *Set*

$$V_i = \bigvee S^{|e_i^\ell|}$$

where $|e_i^\ell|$ is the dimensions of generator e_i^ℓ .

The V_i are constructed so that $(0, \tilde{h}^*(V_i))$ is strongly h^* cohomology isomorphic to $(0, E_i)$. *Strongly h^* cohomology isomorphic* is defined in definition 4.4 and the paragraph before corollary 4.5. By construction the spectral sequence

$$E_r(\widehat{Z}((N(I, \sigma); (\underline{CV}, \underline{V})))$$

is isomorphic to the spectral sequence

$$\left[\bigoplus_{\substack{J \cup L \cup I = [m] \\ L \cup \sigma \in K}} E^J \otimes W^L \right].$$

The spectral sequence converges to $h^*(\widehat{Z}(N(I, \sigma); (\underline{CV}, \underline{V})))$. By the wedge lemma, 2.7

$$h^*(\widehat{Z}(N(I, \sigma); (\underline{CV}, \underline{V}))) = h^*(\Sigma|N(I, \sigma)|) \otimes E^{[m]-I}.$$

Combining these results gives the following calculation of the cohomology of the polyhedral smash product functor, which by the algebraic splitting theorem, 4.6, gives the cohomology of the polyhedral product functor for CW pairs satisfying the freeness condition. The difference between the spectral sequence converging to $h^*(Z(K; (\underline{X}, \underline{A})))$ and the one converging to $\tilde{h}^*(\widehat{Z}(K; (\underline{X}, \underline{A})))$ is describe in the proof of Corollary 4.6.

The extensions in the spectral sequence converging to

$$h^*(\widehat{Z}(K; (\underline{X}, \underline{A})))$$

appear as extension is the spectral sequence converging to $h^*(\Sigma|N(I, \sigma)|) \otimes E^{[m]-I}$. So we may write $E^{[m]-I} \otimes \tilde{h}^*(\Sigma|N(I, \sigma)|)$ below rather than the associated graded group.

Theorem 5.4.

(1)

$$\tilde{h}^*(\widehat{Z}(K; (\underline{X}, \underline{A}))) = \bigoplus_{I \subset [m], \sigma \in K} E^{[m]-I} \otimes \tilde{h}^*(\Sigma|N(I, \sigma)|) \otimes Y^{I, \sigma}.$$

as h^* -modules with $\tilde{h}^*(\Sigma \emptyset) = 1$. The factors, B_i of $Y^{I, \sigma}$ are subsets of the **reduced cohomology** of A_i . i.e. $1 \in B_i$ as defined in Definition 2.2 does not appear.

(2) *Similarly*

$$h^*(Z(K; (\underline{X}, \underline{A}))) = \bigoplus_{I \subset [m], \sigma \in K} E^{[m]-I} \otimes \tilde{h}^*(\Sigma |N(I, \sigma)|) \otimes Y^{I, \sigma}.$$

as h^* -modules with $\tilde{h}^*(\Sigma \emptyset) = 1$, but now the factors, B_i of $Y^{I, \sigma}$ are subsets of the **un-reduced cohomology** of A_i . i.e. 1 as defined in Definition 2.2 may appear in a coordinate of $Y^{I, \sigma}$.

A convenient reformulation of Theorem 5.4 is given in the next corollary.

Corollary 5.5.

$$h^* = \bigoplus_{\substack{I \cap J = \emptyset \\ I \cup J = [m]}} \tilde{h}^*(\Sigma |N_J|) \otimes E^J \otimes h^*(X^I)/(R)$$

where $I = \{i_1, \dots, i_t\}$, $X^I = X_{i_1} \times \dots \times X_{i_t}$, and (R) is the ideal generated by $C^S \subset H^*(X^I)$ where S is not simplex of K .

The reduced Poincare series for $H^*(\widehat{Z}(K; (\underline{X}, \underline{A})))$ is recorded as the following corollary.

Corollary 5.6.

$$\overline{P}(H^*(\widehat{Z}(K; (\underline{X}, \underline{A})))) = \sum_{I, \sigma} t \overline{P}(Y^{I, \sigma}) \times \overline{P}(H^*(|N(I, \sigma)|) \times \overline{P}(E^{[m]-I}).$$

where $\overline{P}(H^*(\emptyset)) = 1/t$.

Remark 5.7. The Poincare series was first computed in [4].

Example 5.8. We illustrate Corollary 5.6 with the example of K a simplicial complex with 3 vertices and edges $\{1, 3\}, \{1, 2\}$. $H^*(X) = \mathbb{Z}\{b_4, c_6\}$, $H^*(A) = \mathbb{Z}\{e_2, b_4\}$. The cases in the example are indexed by the I in Corollary 5.6 starting with the empty set and building up to $I = \{1, 2, 3\}$. For each I there are the sub cases indexed by the simplices $\sigma \subset I$.

- For $I = \emptyset$. the only possible simplex, σ , is the empty set and $N(I, \emptyset)$ is contractible. So there is no contribution to the Poincare series.
- The next case is $I = \{1\}$. There are two possible simplices, namely $\sigma = \emptyset$ and $\sigma = \{1\}$.

(1) $\sigma = \emptyset$.

In this case $Y^{I, \emptyset} = b_4$. and $|N(I, \emptyset)| = |\{\{2\}, \{3\}\}| = S^0$ which contributes

$$t \overline{P}(Y^{I, \sigma}) \overline{P}(H^*(|N(I, \sigma)|)) \overline{P}(E^{[m]-I}) = t(t^4)(t^2)^2 = t^9$$

to the Poincare series.

(2) $\sigma = \{1\}$.

In this case $Y^{I,\sigma} = c_6$. and $|N(I, \sigma)| = |\{2\}, \{3\}| = S^0$. Thus the term

$$t\overline{P}(Y^{I,\sigma})\overline{P}(H^*(|N(I, \sigma)|))\overline{P}(E^{[m]-I}) = t(t^6)(t^2)^2 = t^{11}$$

is contributed to the Poincare series.

- Similarly for $I = \{2\}$ there are the cases
 - (1) $\sigma = \emptyset$ with $N(I, \sigma) =$ the edge $\{1, 3\}$ which is contractible.
 - (2) $\sigma = \{2\}$ with $N(I, \sigma) = \{1\}$ which is also contractible.
- $I = \{3\}$. This is similar to $I = \{2\}$.
 - (1) $\sigma = \emptyset$, $N(I, \sigma) =$ the edge $\{1, 2\}$ which is contractible.
 - (2) $\sigma = \{3\}$, $N(I, \sigma) = \{1\}$ which is contractible.
- For $I = \{1, 2\}$ there are 4 possible simplices
 - (1) $\sigma = \emptyset$ with $N(I, \sigma) = \{3\}$ which is contractible.
 - (2) $\sigma = \{1\}$ with $N(I, \sigma) = \{3\}$ which is contractible.
 - (3) $\sigma = \{2\}$, with $Y^{I,\sigma} = b_4 \otimes c_6$ and $N(I, \sigma) = \emptyset$ which contributes

$$t\overline{P}(Y^{I,\sigma})\overline{P}(H^*(|N(I, \sigma)|))\overline{P}(E^{[m]-I}) = (t^6 t^4)(t^2) = t^{12}.$$

to the Poincare series.

(4) $\sigma = \{1, 2\}$

$Y^{I,\sigma} = c_6 \otimes c_6$, $N(I, \{1, 2\}) = \emptyset$. So this case contributes t^{14} to the Poincare series.

- $I = \{1, 3\}$ is identical to $I = \{1, 2\}$ so we get a contribution of t^{12} and t^{14} to the Poincare series.
- $I = \{2, 3\}$
 - (1) The cases there are 3 possible simplices: $\sigma = \emptyset$, $\{2\}$, and $\sigma = \{3\}$. For all 3 simplices $N(I, \sigma) = \{1\}$ which is contractible.
- Finally $I = \{1, 2, 3\} = K$. For all σ , $N(I, \sigma) = \emptyset$.

The sub simplices of K contribute to the Poincare series as follows:

- (1)
 - (a) $\sigma = \{1\}$, $Y^{I,\sigma} = c_6 \otimes b_4 \otimes b_4$
 - (b) $\sigma = \{2\}$, $Y^{I,\sigma} = b_4 \otimes c_6 \otimes b_4$
 - (c) $\sigma = \{3\}$, $Y^{I,\sigma} = b_4 \otimes b_4 \otimes c_6$
 each contributes t^{14} to \overline{P} .
- (2)
 - (a) $\sigma = \{1, 2\}$, $Y^{I,\sigma} = c_6 \otimes c_6 \otimes b_4$
 - (b) $\sigma = \{1, 3\}$, $Y^{I,\sigma} = c_6 \otimes b_4 \otimes c_6$
 each contributes t^{16} to \overline{P} .

(3) $\sigma = \emptyset, Y^{I,\sigma} = b_4 \otimes b_4 \otimes b_4$, contributes t^{12} to \overline{P}

Adding all the terms we get have the Poincare series for the cohomology of $\widehat{Z}(K; (\underline{X}, \underline{A}))$.

$$\overline{P}(H^*(\widehat{Z}(K; (\underline{X}, \underline{A})))) = t^9 + t^{11} + 3t^{12} + 5t^{14} + 2t^{16}.$$

The next two corollaries describe summands in $h^*(\widehat{Z}(K; (\underline{X}, \underline{A})))$. The summand in Corollary 5.9 depends on the E_i . The summand in corollary 5.11 is natural since C_i is the kernel of the map $H^*(X_i) \rightarrow H^*(A_i)$ which is a functor of the pairs (X_i, A_i) .

Corollary 5.9. $\widetilde{h}^*(\Sigma|K|) \otimes E_1 \otimes \cdots \otimes E_m$ is a summand $\widetilde{h}^*(\widehat{Z}(K; (\underline{X}, \underline{A})))$.

Proof. This corresponds to the summands with $I = \emptyset$ in Theorem 5.4. Specifically if $I = \emptyset$ then $\sigma = \emptyset$ and $N(\emptyset, \emptyset) = K$. \square

Remark 5.10. Corollary 5.9 generalizes [3, Theorem 1.12] which describes the cohomology of $\widehat{Z}(K; (\underline{CX}, \underline{X}))$. In this case $B_i = C_i = 0$ and $E_i = \widetilde{H}^*(X_i)$. The only summand is $I = \emptyset$.

However Theorem 1.12 of [3] is not a consequence of Corollary 5.9 since the complicated bookkeeping involved to evaluate all the differentials in the spectral sequence were subsumed by [3, Theorem 1.12] which was used to prove Theorems 5.4 and therefore Corollary 5.9.

Corollary 5.11. Let $I \subset h^*(X_1 \times \cdots \times X_m)$ be the ideal generated by $C_{i_1} \otimes \cdots \otimes C_{i_q}$ where (i_1, \dots, i_q) is not a simplex in K . Then $h^*(X_1 \times \cdots \times X_m)/(I)$ is a sub-ring of $\widetilde{h}^*(Z(K; (\underline{X}, \underline{A})))$.

Proof. There are the maximal summands, $Y^{[m],\sigma}$.

$$\bigoplus_{\sigma \in K} Y^{[m],\sigma} \simeq h^*(X_1 \times \cdots \times X_m)/(I)$$

as h^* modules. The inclusion,

$$\iota : Z(K; (\underline{X}, \underline{A})) \rightarrow X_1 \times \cdots \times X_m$$

induces a surjection in cohomology onto $\bigoplus_{\sigma \in K} Y^{[m],\sigma}$. To prove corollary 5.11 it suffices to show that I is isomorphic to the kernel of $h^*(\iota)$.

We may write $h^*(X_i) = B_i \oplus C_i$. The tensor product $h^*(X_1) \otimes \cdots \otimes h^*(X_m)$ may now be written as a sum of terms of the form $S_1 \otimes \cdots \otimes S_m$ where S_i is B_i or C_i . The map of rings, $h^*(X_1) \otimes \cdots \otimes h^*(X_m) \rightarrow h^*(Z(K; (\underline{X}, \underline{A})))$ is surjective with kernel given by the ideal I . \square

Remark 5.12. The spectral sequence is natural with respect to maps of pairs

$$(X_i, A_i) \rightarrow (Y_i, D_i)$$

(In fact it is a functor of strong h^* -cohomology maps of pairs as described in Corollary 4.5). However the description of the cohomology of $Z(K; (\underline{X}, \underline{A}))$ given in Theorem 5.4 is not natural. It depends on the choice of splitting of $h^*(X_i)$ as $B_i \oplus C_i$ and splitting of $h^*(A_i)$ as $E_i \oplus B_i$. We now describe how the decomposition in Theorem 5.4 and the naturality of the spectral sequence interact.

To this end suppose there are maps of long exact sequences

$$(5.2) \quad \begin{array}{ccccccc} \cdots & \xleftarrow{\delta} & h^*(A_i) & \leftarrow & h^*(X_i) & \leftarrow & \tilde{h}^*(X_i/A_i) \leftarrow \cdots \\ & & \uparrow g_i & & \uparrow f_i & & \uparrow \\ \cdots & \xleftarrow{\delta} & h^*(D_i) & \leftarrow & h^*(Y_i) & \leftarrow & \tilde{h}^*(Y_i/D_i) \leftarrow \cdots \end{array}$$

Diagram 5.2 induces a map of spectral sequences

$$E_r(Z(K; (\underline{Y}, \underline{D}))) \xrightarrow{(f,g)^*} E_r(Z(K; (\underline{X}, \underline{A})))$$

and a map

$$h^*(Z(K; (\underline{Y}, \underline{D}))) \xrightarrow{\ell} h^*(Z(K; (\underline{X}, \underline{A}))).$$

The map ℓ is now described in terms of the decompositions given by Definition 2.2.

For each i there is a decomposition of the h^* -modules in the the top row

$$h^*(A_i) = E_i \oplus B_i, \quad h^*(X_i) = B_i \oplus C_i.$$

The bottom row has a corresponding decomposition

$$h^*(D_i) = E'_i \oplus B'_i, \quad h^*(Y_i) = B'_i \oplus C'_i.$$

Suppose $\alpha \in h^*(Z(K; (\underline{Y}, \underline{D})))$ is a class appearing in a summand

$$(E')^{[m]-I} \otimes \tilde{h}^*(\Sigma|N(I, \sigma)|) \otimes (Y')^{I, \sigma}$$

of the decomposition of $h^*(Z(K; (\underline{Y}, \underline{D})))$ given by Theorem 5.4.

Specifically

$$(5.3) \quad \alpha = \bigotimes_J (e'_j) \otimes n \otimes \bigotimes_{|\sigma|} (c'_s) \otimes \bigotimes_L (b'_\ell)$$

where $I = L \cup |\sigma|$, $J = [m] - I$ and $n \in \tilde{h}^*(\Sigma|N(I, \sigma)|)$

In order to compute $\ell(\alpha) \in h^*(Z(K; (\underline{X}, \underline{A})))$ we note that the decompositions of $h^*(A_i)$ and $h^*(X_i)$ into E_i, B_i and C_i imply unique representations

$$g_i(e'_i) = e_i + \bar{b}_i, \quad f_i(b'_i) = b_i + \bar{c}_i \text{ and } f_i(c'_i) = c_i$$

where $e_i \in E_i$, b_i and $\bar{b}_i \in B_i$, \bar{c}_i and $c_i \in C_i$.

Formally substitute $e_i + \bar{b}_i$ for e'_i , $b_i + \bar{c}_i$ for b'_i and c_i for c'_i in (5.3). The resulting expression is a sum of terms with factors e_i , b_i , \bar{b}_i , \bar{c}_i , c_i and $n \in \tilde{h}^*(\Sigma|N(I, \sigma)|)$. Each summand determines a summand in $\ell(\alpha)$. There are a number of cases.

The easiest case is the summand without any over-lined factors. For this term the map of spectral sequences $(f, g)^*$ respects the decomposition of Definition 2.2 at the E_1 page and contributes the summand

$$\bigotimes_J (e_j) \otimes n \otimes \bigotimes_{|\sigma|} (c_s) \bigotimes_L (b_\ell)$$

to $\ell(\alpha)$.

Now suppose there are terms of the formal sum with non-zero \bar{b}_i factors i.e. there is an indexing set $Q \subset J$ where \bar{b}_q is not zero for $q \in Q$. In this situation there are formal summands in

$$E^{J \setminus Q} \otimes n \otimes C^{|\sigma|} \otimes B^{L \cup Q}.$$

In terms of the decomposition of Theorem 5.4 these terms contribute classes in

$$E^{J \setminus Q} \otimes \tilde{h}^*(\Sigma|N(I \cup Q, \sigma)|) \otimes C^{|\sigma|} \otimes B^{L \cup Q}$$

to $\ell(\alpha)$ (recall that $I = L \cup |\sigma|$).

The simplicial complex $N(I \cup Q, \sigma)$ is a sub-simplicial complex of $N(I, \sigma)$. To prove this suppose $\tau \in N(I \cup Q, \sigma)$ then $\tau \cup \sigma \in K$ and $|\tau| \subset J \setminus Q \subset J$ so $\tau \in N(I, \sigma)$.

The formal summand

$$E^{J \setminus Q} \otimes n \otimes C^{|\sigma|} \otimes B^{L \cup Q}.$$

contributes

$$E^{J \setminus Q} \otimes \iota^*(n) \otimes C^{|\sigma|} \otimes B^{L \cup Q}.$$

to $\ell(\alpha)$ where ι^* is induced by the inclusion

$$\iota^* : \tilde{h}^*(\Sigma|N(I, \sigma)|) \rightarrow \tilde{h}^*(\Sigma|N(I \cup Q, \sigma)|).$$

Indeed ι^* at the cochain level is the map which sends the dual of a simplex, τ to zero if τ is not a simplex in $N(I \cup Q, \sigma)$ and to the dual of τ if τ is a simplex in $N(I \cup Q, \sigma)$. This agrees with the map $(f, g)^*$.

Finally suppose there are terms of the formal sum with non-zero \bar{c}_i factors. i.e there are indexing sets $P \subset L$ where $\bar{c}_p \neq 0$ for $p \in P$. These formal summands are classes in

$$E^J \otimes n \otimes C^{|\sigma| \cup P} \otimes B^{L \setminus P}.$$

In terms of the decomposition of Theorem 5.4 these terms will contribute classes in

$$E^J \otimes \tilde{h}^*(\Sigma|N(I, \sigma \cup P)|) \otimes C^{|\sigma| \cup P} \otimes B^{L \setminus P}.$$

We shown that $N(I, \sigma \cup P)$ is a sub complex of $N(I, \sigma)$ if $\sigma \cup P$ is a simplex in K (otherwise thie summand lies in the zero group). Suppose $\sigma \cup P$ is a simplex in K , say τ . Let $\rho \in N(I, \tau)$. Then ρ has vertices in J and $\rho \cup \sigma \cup P$ is a simplex in K which implies $\rho \cup \sigma$ is also a simplex in K and $\rho \in N(I, \sigma)$. The summands of $\ell(\alpha)$ with $\bar{\tau}_p$ factors are represented by classes which jump filtration in the spectral sequence with n replaced by $\iota^*(n)$ ($\iota : N(I, \tau) \rightarrow N(I, \sigma)$ the inclusion).

This description of ℓ will be applied in (6.5).

6. Products

The purpose of this section is to describe the ring structure in $H^*(Z(K; (\underline{X}, \underline{A})), \mathbb{R})$ with \mathbb{R} a commutative ring. (there are similar results for $h^*(Z(K; (\underline{X}, \underline{A})))$). As in the previous sections (X_i, A_i) are assumed to be based CW pairs which satisfy the freeness condition of Definition 2.2. We write $H^*(Z(K; (\underline{X}, \underline{A})))$ for $H^*(Z(K; (\underline{X}, \underline{A})), \mathbb{R})$ in the sequel.

Recall the computation of $H^*(Z(K; (\underline{X}, \underline{A})))$ as an h^* -module given by 5.4.

$$h^*(Z(K; (\underline{X}, \underline{A}))) = \bigoplus_{I \subset [m], \sigma \in K} E^{[m]-I} \otimes \tilde{h}^*(\Sigma |N(I, \sigma)|) \otimes Y^{I, \sigma}.$$

In particular $h^*(Z(K; (\underline{X}, \underline{A})))$ is generated, as an h^* -module by monomials

$$n \otimes a_1 \otimes \cdots \otimes a_m$$

with $a_i \in E_i, B_i$ or C_i and

$$n \in \begin{cases} \tilde{H}^*(\Sigma |N(I, \sigma)|) & \text{if } \sigma = \{i | a_i \in C_i\} \subset [m] \text{ is a simplex in } K \\ 0 & \text{otherwise.} \end{cases}$$

where $I = \{i | a_i \in C_i \text{ or } B_i\} \subset [m]$ and $N(I, \sigma)$ is defined in Definition 5.2.

To describe

$$H^*(Z(K; (\underline{X}, \underline{A})))$$

as a ring it suffices to define a paring on the summands of Theorem 5.4

$$\left[E^{[m]-I_1} \otimes \tilde{H}^*(\Sigma |N(I_1, \sigma_1)|) \otimes Y^{I_1, \sigma_1} \right] \otimes \left[E^{[m]-I_2} \otimes \tilde{H}^*(\Sigma |N(I_2, \sigma_2)|) \otimes Y^{I_2, \sigma_2} \right] \xrightarrow{\cup} H^*(Z(K; (\underline{X}, \underline{A})))$$

Specifically suppose that

$$\alpha = n_\alpha \otimes a_1 \otimes \cdots \otimes a_m \in E^{[m]-I_1} \otimes \tilde{H}^*(\Sigma|N(I_1, \sigma_1)|) \otimes Y^{I_1, \sigma_1}$$

where $n_\alpha \in \tilde{H}^*(\Sigma|N(I_1, \sigma_1)|)$ and $a_i \in E_i, B_i$ or C_i .

$$\gamma = n_\gamma \otimes g_1 \otimes \cdots \otimes g_m \in E^{[m]-I_2} \otimes \tilde{H}^*(\Sigma|N(I_2, \sigma_2)|) \otimes Y^{I_2, \sigma_2}$$

where $n_\gamma \in \tilde{H}^*(\Sigma|N(I_2, \sigma_2)|)$ and $g_i \in E_i, B_i$ or C_i .

We will describe $\alpha \cup \gamma$ in terms of a coordinate wise multiplication of a_i and g_i and a paring

$$(6.1) \quad H^*(\Sigma|N(I_1, \sigma_1)|) \otimes H^*(\Sigma|N(I_2, \sigma_2)|) \rightarrow H^*(\Sigma|N(I_3, \sigma_3)|)$$

where (I_3, σ_3) will be defined in terms of (I_1, σ_1) and (I_2, σ_2)

The pairing, (6.1) will be defined in terms of the $*$ -product defined in [3] which we now recall.

Writing $Z(K)$ for $Z(K; (\underline{X}, \underline{A}))$ and $\widehat{Z}(K_I)$ for $\widehat{Z}(K_I; (\underline{X}, \underline{A})_I)$, *partial diagonals*

$$\Delta_I^{J,L} : \widehat{Z}(K_I) \rightarrow \widehat{Z}(K_J) \wedge \widehat{Z}(K_L)$$

$(J \cup L = I)$ are defined which fit into a diagram

$$\begin{array}{ccc} Z(K) & \xrightarrow{\widehat{\Delta}} & Z(K) \wedge Z(K) \\ \Pi \downarrow & & \downarrow \Pi \wedge \Pi \\ \widehat{Z}(K_I) & \xrightarrow{\Delta_I^{J,L}} & \widehat{Z}(K_J) \wedge \widehat{Z}(K_L). \end{array}$$

where $\widehat{\Delta}$ is the reduced diagonal and Π is the projection.

The definition of the partial diagonals and projections are as follows (the notation is as in Definitions 2.3 and 2.4).

(1) For $I \subset [m]$ and $\sigma \in K$ there is the projection followed by the collapsing map

$$\pi : D(\sigma) \rightarrow D(\sigma \cap I) \rightarrow \widehat{D}(\sigma \cap I).$$

These composites are compatible with the maps in the colimit and induce the vertical maps

$$\Pi : Z(K) \rightarrow \widehat{Z}(K_I).$$

(2) Let $W_I^{J,L}$ be defined by

$$W_I^{J,L} = \begin{cases} Y_i & \text{if } i \in I - J \cap L \\ Y_i \wedge Y_i & \text{if } i \in J \cap L \end{cases}$$

where Y_i is either X_i or A_i as in Definition 2.4.

There is a homoemorphism

$$Sh : W_I^{J,L} \rightarrow \widehat{Y}^J \wedge \widehat{Y}^L$$

given by the evident shuffle which is compatible with the maps in the colimit.

(3) Define

$$Y^I \rightarrow \widehat{Y}^J \wedge \widehat{Y}^L$$

by first mapping into $W_I^{J,L}$ by

- (a) the identity of Y_i if $i \in I - J \cap L$ or
- (b) the diagonal of Y_i if $i \in J \cap L$

followed by Sh . The maps induce a map of colimits which define the partial diagonal, $\Delta_I^{J,L}$.

Given cohomology classes $u \in H^p(\widehat{Z}(K_J))$, $v \in H^q(\widehat{Z}(K_L))$, the \ast -product is defined by

$$(6.2) \quad u \ast v = (\Delta_I^{J,L})^*(u \otimes v) \in H^{p+q}(\widehat{Z}(K_I)).$$

In [3] the ring structure of $H^*(Z(K, (\underline{X}, \underline{A})))$ is shown to be induced by the \ast -product.

The special case of $(\underline{X}, \underline{A}) = (D^1, S^0)$ is particularly important.

The splitting of Theorem, 2.6, and Theorem 2.7 imply there are homotopy equivalences

$$\Sigma Z(K; (D^1, S^0)) \rightarrow \bigvee_{I \subset [m]} \Sigma \widehat{Z}(K_I; (D^1, S^0))$$

and

$$\widehat{Z}(K_I; (D^1, S^0)) \xrightarrow{\sim} |K_I| \ast (\widehat{S}^0)^I \simeq \Sigma |K_I|.$$

With $I = J \cup L$ a pairing

$$(6.3) \quad H^*(\Sigma |K_J|) \otimes H^*(\Sigma |K_L|) \rightarrow H^*(\Sigma |K_I|)$$

is induced by the partial diagonals map $\Delta_I^{J,L}$.

$$\Sigma|K_I| \simeq \widehat{Z}(K_I; (D^1, S^0)) \xrightarrow{\Delta_I^{J,L}} \widehat{Z}(K_J; (D^1, S^0)) \wedge \widehat{Z}(K_L; (D^1, S^0)) \simeq \Sigma|K_J| \wedge \Sigma|K_L|.$$

Theorem 6.1. *The product*

$$\left[n_\alpha \otimes a_1 \otimes \cdots \otimes a_m \right] \cup \left[n_\gamma \otimes g_1 \otimes \cdots \otimes g_m \right] \in H^*(Z(K; (\underline{X}, \underline{A})))$$

is given by the $*$ -product of n_α and n_β composed with an inclusion map described in Lemma 6.3 and a coordinate wise product defined as follows:

- (1) If $a_i, g_i \in H^*(X_i)$ the product in the i -th coordinate is the product in $H^*(X_i)$.
- (2) If $a_i, g_i \in H^*(A_i)$ the product is induced by the product in $H^*(A_i)$.
- (3) if $a_i \in E_i, g_i \in C_i$ or $g_i \in E_i, a_i \in C_i$ the product is zero.

The rest of this section is devoted to proving Theorem 6.1.

A description of the diagonal map

$$Z(K; (\underline{X}, \underline{A})) \rightarrow Z(K; (\underline{X}, \underline{A})) \times Z(K; (\underline{X}, \underline{A}))$$

and the partial diagonal maps

$$\widehat{Z}(K_I; (\underline{X}, \underline{A})_I) \rightarrow \widehat{Z}(K_J; (\underline{X}, \underline{A})_J) \wedge \widehat{Z}(K_L; (\underline{X}, \underline{A})_L)$$

are now given. The description uses the following lemma where the notation

$$[(\underline{X}, \underline{A})_I^{J,L}]_i = \begin{cases} (X_i, A_i) & \text{if } i \in I - J \cap L \\ (X_i \wedge X_i, A_i \wedge A_i) & \text{if } i \in J \cap L \end{cases}$$

is used in part (c).

Lemma 6.2. (a) Suppose $A_m \subset X_m, B_m \subset Y_m$ then there is a natural map

$$Sh : Z(K; (\underline{X} \times \underline{Y}, \underline{A} \times \underline{B})) \rightarrow Z(K; (\underline{X}, \underline{A})) \times Z(K; (\underline{Y}, \underline{B}))$$

where

$$(\underline{X} \times \underline{Y}, \underline{A} \times \underline{B}) = \{(X_m \times Y_m, A_m \times B_m)\}.$$

(b) The diagonal maps of pairs, $(X_m, A_m) \rightarrow (X_m \times X_m, A_m \times A_m)$ defines a map

$$\tilde{\Delta} : (\underline{X}, \underline{A}) \rightarrow (\underline{X} \times \underline{X}, \underline{A} \times \underline{A})$$

which induces a map of spectral sequences.

$$E_r(Z(K; (\underline{X} \times \underline{X}, \underline{A} \times \underline{A}))) \rightarrow E_r(Z(K; (\underline{X}, \underline{A})))$$

The diagonal map is the composite

$$\Delta : Z(K; (\underline{X}, \underline{A})) \xrightarrow{\tilde{\Delta}} Z(K; (\underline{X} \times \underline{X}, \underline{A} \times \underline{A})) \xrightarrow{Sh} Z(K; (\underline{X}, \underline{A})) \times Z(K; (\underline{X}, \underline{A})).$$

(c) The partial diagonal map of pairs $\hat{\Delta}_I^{J,L} : (\underline{X}, \underline{A}) \rightarrow (\underline{X}, \underline{A})_I^{J,L}$

$$\hat{\Delta}_I^{J,L} = \begin{cases} \text{the identity} & \text{if } i \in I - J \cap L \\ \text{the reduced diagonal} & \text{if } i \in J \cap L. \end{cases}$$

induces a map of spectral sequences.

$$E_r(\hat{Z}(K_I; (\underline{X}, \underline{A})_I^{J,L})) \rightarrow E_r(\hat{Z}(K_I; (\underline{X}, \underline{A})_I))$$

$\Delta_I^{J,L}$ is the composite

$$\hat{Z}(K_I; (\underline{X}, \underline{A})_I) \xrightarrow{\hat{\Delta}_I^{J,L}} \hat{Z}(K_I; (\underline{X}, \underline{A})_I^{J,L}) \xrightarrow{Sh} \hat{Z}(K_J; (\underline{X}, \underline{A})_J) \wedge \hat{Z}(K_L; (\underline{X}, \underline{A})_L).$$

Proof. (a): From the definition of the polyhedral product functor, Definition 2.3, $Z(K; (\underline{X} \times \underline{Y}, \underline{A} \times \underline{B}))$ is a colimit of spaces, $D(\sigma, (\underline{X} \times \underline{Y}, \underline{A} \times \underline{B}))$. By shuffling the factors, there is a map $D(\sigma, (\underline{X} \times \underline{Y}, \underline{A} \times \underline{B})) \rightarrow D(\sigma, (\underline{X}, \underline{A})) \times D(\sigma, (\underline{Y}, \underline{B}))$. The maps are compatible with the maps into the spaces of the colimit defining $Z(K; (\underline{X}, \underline{A})) \times Z(K; (\underline{Y}, \underline{B}))$, proving (a).

(b): The diagonal

$$\Delta : Z(K; (\underline{X}, \underline{A})) \rightarrow Z(K; (\underline{X}, \underline{A})) \times Z(K; (\underline{X}, \underline{A}))$$

is induced at the level of $D(\sigma)$ by $\tilde{\Delta}$ followed by a shuffle. It follows from (a) that Δ factors as indicated.

(c): Is similar.

□

Since the product in $H^*(Z(K; (\underline{X}, \underline{A})))$ is determined by the $*$ -product, as in (6.2) it suffices to compute the map induced by $\Delta_I^{J,L}$. It follows from Lemma 6.2 that $\Delta_I^{J,L}$ decomposes as the composition of the map induced by the shuffle followed by the map induced by

$\widehat{\Delta}_I^{J,L}$. The shuffle map may be computed using the fact that the spectral sequence, and hence $H^*(\widehat{Z}(K;(\underline{X}, \underline{A})))$ is a functor of the strong H^* -cohomology type (Corollary 4.5) and the decomposition in Definition 2.2. The map induced by $\widehat{\Delta}_I^{J,L}$ is computed using the naturally discussed in remark 5.12.

We first describe $(\widehat{\Delta}_I^{J,L})^*$. To this end the decomposition of Definition 2.2 is described for the pair $(\underline{X} \wedge \underline{X}, \underline{A} \wedge \underline{A})$.

There is the long exact sequence

$$\xleftarrow{\delta} \widetilde{H}^*(A_i \wedge A_i) \xleftarrow{i} \widetilde{H}^*(X_i \wedge X_i) \leftarrow \widetilde{H}^*(X \wedge X / A \wedge A) \leftarrow$$

The image, kernel and cokernel of Definition 2.2 for the above exact sequence will be denoted $\widehat{B}_i, \widehat{C}_i$ and \widehat{E}_i respectively. In terms of B_i, C_i, E_i associated to the pair (X_i, A_i)

$$(6.4) \quad \begin{aligned} \widehat{B}_i &= B_i \otimes B_i \\ \widehat{C}_i &= \left(C_i \otimes C_i \right) \oplus \left(B_i \otimes C_i \right) \oplus \left(C_i \otimes B_i \right) \\ \widehat{E}_i &= \left(E_i \otimes E_i \right) \oplus \left(E_i \otimes B_i \right) \oplus \left(B_i \otimes E_i \right) \end{aligned}$$

The diagonal induces the product in $H^*(X_i)$ and $H^*(A_i)$

$$(6.5) \quad \begin{aligned} \widehat{B}_i &\rightarrow H^*(X_i) = B_i \oplus C_i \\ \widehat{C}_i &\rightarrow C_i \\ \widehat{E}_i &\rightarrow H^*(A_i) = E_i \oplus B_i \end{aligned}$$

The map of long exact sequences induced by the diagonal

$$(6.6) \quad \begin{array}{ccccccc} \cdots & \xleftarrow{\delta} & h^*(A_i) & \leftarrow & h^*(X_i) & \leftarrow & \widetilde{h}^*(X_i/A_i) & \leftarrow \cdots \\ & & \uparrow \Delta_i^* & & \uparrow \Delta_i^* & & \uparrow & \\ \cdots & \xleftarrow{\delta} & h^*(A_i \wedge A_i) & \leftarrow & h^*(X_i \wedge X_i) & \leftarrow & \widetilde{h}^*(X_i \wedge X_i / A_i \wedge A_i) & \leftarrow \cdots \end{array}$$

is an example of the more general situation described at the end of Section 5.

The map

$$\widehat{\Delta}_I^{J,L} : \widehat{Z}(K_I;(\underline{X}, \underline{A})_I) \rightarrow \widehat{Z}(K_I;(\underline{X}, \underline{A})_I^{J,L})$$

is induced by a map of pairs $(\underline{X}, \underline{A})_I \rightarrow (\underline{X}, \underline{A})_I^{J,L}$ which at E_∞ is the product described in (6.5).

The details follow. We first have to adjust the indexing sets in Theorem 5.4. The vertex set of the simplicial complex K_I is I not $[m]$. Since I now denotes the vertex set we cannot use it in the notation for the factor $Y^{I,\sigma}$. We replace I in Theorem 5.4 with F . With these modifications

$$H^*(\widehat{Z}(K_I; (\underline{X}, \underline{A})_I^{J,L}))$$

is a sum of groups

$$\bigoplus_{F \subset I, \sigma \in K_I} \overline{E}^{I-F} \otimes \widetilde{H}^*(\Sigma|N(F, \sigma)|) \otimes \overline{Y}^{F,\sigma}$$

where a factor \overline{E}_i of \overline{E}^{I-F} is \widehat{E}_i if $i \in J \cap L$ and E_i otherwise. The factors of $\overline{Y}^{F,\sigma}$ are defined analogously. Specifically a factor \overline{C}_i of $\overline{Y}^{F,\sigma}$ is \widehat{C}_i if $i \in J \cap L$ and C_i otherwise. A factor \overline{B}_i of $\overline{Y}^{F,\sigma}$ is \widehat{B}_i if $i \in J \cap L$ and is B_i otherwise.

Similarly

$$H^*(\widehat{Z}(K_I; (\underline{X}, \underline{A})_I))$$

is a sum

$$\bigoplus_{F' \subset I, \tau \in K_I} E^{I-F'} \otimes \widetilde{H}^*(\Sigma|N(F', \tau)|) \otimes Y^{F',\tau}.$$

The map

$$\widehat{\Delta}_I^{J,L} : \widehat{Z}(K_I; (\underline{X}, \underline{A})_I) \rightarrow \widehat{Z}(K_I; (\underline{X}, \underline{A})_I^{J,L})$$

induces a map in cohomology which restricted to each summand is a map

$$\overline{E}^{I-F} \otimes \widetilde{H}^*(\Sigma|N(F, \sigma)|) \otimes \overline{Y}^{F,\sigma} \rightarrow \bigoplus_{F' \subset I, \tau \in K_I} E^{I-F'} \otimes \widetilde{H}^*(\Sigma|N(F', \tau)|) \otimes Y^{F',\tau}.$$

This map is computed using the naturality discussion at the end of Section 5 with maps on \widehat{E}_i , \widehat{B}_i , and \widehat{C}_i given by (6.5). Specifically the map induced by the product on \widehat{E}_i may have summands in B_i thus enlarging F to a larger indexing set, F' . Also some of the terms \widehat{B}_i which appear in $\overline{Y}^{F,\sigma}$ map via the product to summands with a factor of C_i enlarging the simplex σ to τ . Thus we have proven lemma 6.3 below. The sum is over all $F' \subset I$ and τ .

Lemma 6.3.

$$(\widehat{\Delta}_I^{J,L})^* : \overline{E}^{I-F} \otimes \widetilde{H}^*(\Sigma|N(F, \sigma)|) \otimes \overline{Y}^{F, \sigma} \rightarrow \bigoplus_{F', \tau} E^{I-F'} \otimes \widetilde{H}^*(\Sigma|N(F', \tau)|) \otimes Y^{F', \tau},$$

where $F' \supset F, \tau \supset \sigma$, the product, (6.5), induces the maps on the factors in \overline{E} and \overline{Y} and $\widetilde{H}^*(\Sigma|N(F, \sigma)|) \rightarrow \widetilde{H}^*(\Sigma|N(F', \tau)|)$ is induced by the inclusions

$$N(F', \tau) \rightarrow N(F, \sigma).$$

□

Intuitively the diagonal induces the product on the coordinates. Because of the mixing of E 's, B 's and C 's the cohomology of the links map to the cohomology of the resulting sub-links .

Note that (6.4) implies neither $E_i \otimes C_i$ nor $C_i \otimes E_i$ appear in $H^*(\widehat{Z}(K_I; (\underline{X}, \underline{A})_I^{J,L}))$. This implies the product of classes in $Z(K; (\underline{X}, \underline{A}))$ involving C_i and E_i must be zero.

An important special case is that of wedge decomposable spaces.

Definition 6.4. A collection of spaces, $(\underline{X}, \underline{A})$ is wedge decomposable if for all i

$$X_i = B_i \vee C_i$$

and

$$A_i = B_i \vee E_i$$

where

$$E_i \rightarrow B_i \vee C_i$$

is null homotopic.

In this case there is none of the mixing of the products complicating the map in Lemma 6.3.

Corollary 6.5. If $(\underline{X}, \underline{A})$ is wedge decomposable then the product of

$$\left[n_\alpha \otimes a_1 \otimes \cdots \otimes a_m \right] \cup \left[n_\gamma \otimes g_1 \otimes \cdots \otimes g_m \right] \in H^*(Z(K; (\underline{X}, \underline{A})))$$

is given by the $*$ -product of n_α and n_β and a coordinate wise product defined as follows:

- (1) If $a_i, g_i \in H^*(B_i)$ the product in the i -th coordinate is the product in $H^*(B_i)$.
- (2) If $a_i, g_i \in H^*(C_i)$ the product is induced by the product in $H^*(C_i)$.

- (3) If $a_i, g_i \in H^*(E_i)$ the product is induced by the product in $H^*(E_i)$.
 (4) The product is zero otherwise.

Example 6.6. We illustrate Lemma 6.3 for $H^*(\widehat{Z}(K; (D^1, S^0)))$.

E_i is generated by a zero dimensional class, t_0 for all i . The product $E_i \otimes E_i \rightarrow E_i$ is an isomorphism ($t_0 \otimes t_0 \mapsto t_0$). In particular $F = F'$.

In this case the pair $(D^1 \wedge D^1, S^0 \wedge S^0)$ is homotopy equivalent to (D^1, S^0) and $\widehat{\Delta}_I^{J,L}$ is a homotopy equivalence.

As a consequence the $*$ -product, (6.3)

$$H^*(\Sigma|K_J|) \otimes H^*(\Sigma|K_L|) \rightarrow H^*(\Sigma|K_I|)$$

is completely determined by the map induce by the shuffle

$$Sh^* : \widetilde{H}^*(\widehat{Z}(K_J; (D^1, S^0)_J)) \otimes \widetilde{H}^*(\widehat{Z}(K_L; (D^1, S^0)_L)) \rightarrow \widetilde{H}^*(\widehat{Z}(K_I; (D^1, S^0)_I^{J,L})).$$

In order to describe the shuffle map for general $(\underline{X}, \underline{A})$ satisfying the freeness condition it is convenient to give yet another description of the spectral sequence. Motivated by the proof of Theorem 5.4 we introduce variables, t_i and s_i into the E_1 term of the spectral sequence. The degree of $t_i = 0$ and the degree of $s_i = 1$.

For K a simplicial complex with vertex set $[m]$ recall the E_1 term is a sum of groups

$$\widetilde{H}^*(X_i/A_i)^\tau \otimes \widetilde{H}^*(\widehat{A}_i)^{\tau^c}$$

(τ a simplex of K). Which in turn is a sum of groups

$$(6.7) \quad C^\sigma \otimes W^{\sigma'} \otimes E^P \otimes B^Q$$

Where σ and σ' are simplices of K whose union is is a simplex $\tau \in K$ and $P \cup Q = \tau^c$. In the notation of Theorem 5.4, $Q \cup |\sigma| = I$, $P \cup |\sigma'| = [m] - I$. The filtration of this summand is the weight of the simplex τ in the left lexicographical ordering of the simplices.

Replace E_i with $t_i E_i$ and W_i with $s_i E_i$ in (6.7) and arrange the sum as follows

$$(6.8) \quad E_1 = \bigoplus_{I,J,\sigma} \left(\bigoplus_{P,\sigma'} E^J \otimes Y^{I,\sigma} t^P s^{\sigma'} \right) \\ I \cup J = [m], I \cap J = \emptyset, \sigma \in K_I \quad P \cup |\sigma'| = J, \sigma \cup \sigma' \in K$$

Define a differential on E_1 by $\delta t_i = s_i$, $\delta s_i = 0$. The proof of Theorem 5.4 implies the following proposition.

Proposition 6.7. (1) *The spectral sequence*

$$E_r(\widehat{Z}(K; (\underline{X}, \underline{A})) \Rightarrow H^*(\widehat{Z}(K; (\underline{X}, \underline{A})))$$

is isomorphic to the spectral sequence with E_1 term

$$\bigoplus_{I,J,\sigma} \left(\bigoplus_{P,\sigma'} E^J \otimes Y^{I,\sigma} t^P s^{\sigma'} \right) \\ I \cup J = [m], I \cap J = \emptyset, \sigma \in K_I \quad P \cup |\sigma'| = J, \sigma \cup \sigma' \in K$$

and differential $\delta t_i = s_i$.

(2) *Let $N(I, \sigma)$ be the simplicial complex defined in Definition 5.2. Then*

$$\bigoplus_{P,\sigma'} E^J \otimes Y^{I,\sigma} t^P s^{\sigma'} \\ P \cup |\sigma'| = J, \sigma \cup \sigma' \in K$$

is isomorphic to

$$(E^J \otimes Y^{I,\sigma}) \otimes E_1(\widehat{Z}(N(I, \sigma); (D^1, S^0)))$$

as differential groups.

It will be convenient to write N_J for $|N(I, \sigma)|$. So the summands of $H^*(\widehat{Z}(K; (\underline{X}, \underline{A})))$ have the form

$$E^J \otimes H^*(\Sigma|N_J|) \otimes Y^{I,\sigma}.$$

The shuffle map is induced by

$$\widehat{D}(\sigma_I) \rightarrow \widehat{D}(\sigma_J) \wedge \widehat{D}(\sigma_L)$$

where $\sigma_J = \sigma_I \cap J$, $\sigma_L = \sigma_I \cap L$. This map induces the shuffle on the E, C and B terms and induces the shuffle on the t_i, p_i terms. Specifically the shuffle map induces a map

$$Sh^* : (E^{J_1} \otimes Y^{I_1, \sigma_1} t^{P_1} s^{\sigma'_1}) \otimes (E^{J_2} \otimes Y^{I_2, \sigma_2} t^{P_2} s^{\sigma'_2}) \rightarrow \bigoplus \overline{E}^J \otimes \overline{Y}^{I, \sigma} t^P s^{\sigma'}$$

with $P_1 \cup |\sigma'_1| = J_1, P_2 \cup |\sigma'_2| = J_2$. \overline{E} and \overline{Y} are define in the paragraph before Lemma 6.3 and $P \cup |\sigma'| = J$.

From (6.4) it follows that Sh^* is zero if there is a coordinate, i with E_i a factor in $E^{J_1} \otimes Y^{I_1, \sigma_1} t^{P_1} s^{\sigma'_1}$ and a factor C_i in $E^{J_2} \otimes Y^{I_2, \sigma_2} t^{P_2} s^{\sigma'_2}$ (or visa-verse). Hence there is no loss of generality to assume

$$(6.9) \quad J_1 \cap |\sigma_2| = \emptyset = J_2 \cap |\sigma_1|$$

Next notice that Sh^* takes any coordinate with a factor of E_i to \overline{E}_i , any coordinate involving C_i to \overline{C}_i and coordinates with both factors in B_i to \overline{B}_i .

Hence the image of Sh^* takes values in the summand

$$\overline{E}^{J_1 \cup J_2} \otimes \overline{Y}^{I_1 \cap I_2, \sigma_1 \cup \sigma_2} t^P s^{\sigma'}$$

where $P \cup |\sigma'| = J_1 \cup J_2$ and $\sigma' \cup \sigma_1 \cup \sigma_2$ is a simplex in K . In particular Sh^* is zero on factors were $\sigma_1 \cup \sigma_2$ is not a simplex in K . It is a consequence of (6.9) that $|\sigma_1 \cup \sigma_2| \subset I_1 \cap I_2$.

Next observe that the shuffle map on t_i and s_i is exactly the map induced by the shuffle map for

$$H^*(\Sigma|N_{J_1}|) \otimes H^*(\Sigma|N_{J_2}|) \rightarrow H^*(\Sigma|N_{J=J_1 \cup J_2}|)$$

Which by example 6.6, is the $*$ -product. Using work of Cai [6] we will give a chain level formula for the $*$ -product in section 7.

Thus Sh^* is the shuffle map on the E, B and C factors and the $*$ -product on the cohomology of the links. Lemma 6.3 and the computation of Sh^* prove Theorem 6.1.

We have shown that the map induced by the shuffle only depends on the ring structure of the cohomology of the subcomplexes of K and the strong homology type of $H^*(X_i)$ and $H^*(A_i)$. The contributions of the ring structure of $H^*(X_i)$ and $H^*(A_i)$ to $H^*(\widehat{Z}(K; (\underline{X}, \underline{A})))$ appear in the diagonal map, $\widehat{\Delta}_I^{J, L}$. So decomposing the partial diagonal into the diagonal composed with the shuffle separates the combinatorial contribution to the ring structure of $Z(K; (\underline{X}, \underline{A}))$ from the cup product structure of the cohomology of $(\underline{X}, \underline{A})$.

Remark 6.8. (1) Q. Zheng, [12], also describes a product in $H^*(Z(K; (\underline{X}, \underline{A})))$.

(2) Proposition 6.7 generalizes to a multiplicative cohomology theory, h^* , satisfying the flatness condition.

7. The Cohomology of $Z(K; (\underline{D}^1, \underline{S}^0))$

Assuming suitable freeness conditions the results of Section 6 determine the cohomology ring $H^*(Z(K; (\underline{X}, \underline{A})))$ in terms of the rings $H^*(X_i)$, $H^*(A_i)$ and the star product on the cohomology of the links. The goal of this section is to complete this description by describing the star product.

Recall that the product

$$H^*(Z(K; (\underline{X}, \underline{A}))) \otimes H^*(Z(K; (\underline{X}, \underline{A}))) \rightarrow H^*(Z(K; (\underline{X}, \underline{A})))$$

is the composite of the map induced by the shuffle

$$Sh^* : H^*(Z(K; (\underline{X}, \underline{A}))) \otimes H^*(Z(K; (\underline{X}, \underline{A}))) \rightarrow H^*(Z(K; (\underline{X} \times \underline{X}, \underline{A} \times \underline{A})))$$

and the map induced by the diagonal

$$\Delta^* : H^*(Z(K; (\underline{X} \times \underline{X}, \underline{A} \times \underline{A}))) \rightarrow H^*(Z(K; (\underline{X}, \underline{A}))).$$

On the summands of Theorem 5.4 it was shown in section 6 that the shuffle has the form

$$(7.1) \quad Sh^* : (E^{J_1} \otimes H^*(\Sigma|N_{J_1}|)) \otimes Y^{I_1, \sigma_1} \otimes (E^{J_2} \otimes H^*(\Sigma|N_{J_2}|) \otimes Y^{I_2, \sigma_2}) \\ \rightarrow \overline{E}^{J_1 \cup J_2} \otimes H^*(\Sigma|N_{J_1 \cup J_2}|) \otimes \overline{Y}^{I_1 \cap I_2, \sigma_1 \cup \sigma_2}$$

where the pairing

$$(7.2) \quad H^*(\Sigma|N_{J_1}|) \otimes H^*(\Sigma|N_{J_2}|) \rightarrow H^*(\Sigma|N_{J_1 \cup J_2}|)$$

is induced by the $*$ -product.

In section 6 it was also shown that the diagonal has the form

$$\overline{E}^{J_1 \cup J_2} \otimes H^*(\Sigma|N_{J_1 \cup J_2}|) \otimes \overline{Y}^{I_1 \cap I_2, \sigma_1 \cup \sigma_2} \rightarrow E^{J'} \otimes H^*(\Sigma|N_{J'}|) \otimes Y^{F', \tau}$$

with the map on the link

$$(7.3) \quad H^*(\Sigma|N_{J_1 \cup J_2}|) \rightarrow H^*(\Sigma|N_{J'}|)$$

induced by an inclusion $\iota : N_{J'} \rightarrow N_{J_1 \cup J_2}$.

We describe the $*$ -product and inclusion map on the links by constructing a filtered chain complex for $H^*(Z(K; (D^1, S^0)))$. The spectral sequence associated to this filtered complex is the spectral sequence $E_r(Z(K; (D^1, S^0)))$.

For $I \subset [m]$ recall that there are classes, s_i of degree 1 and t_i of degree 0 such that the E_1 page of the spectral sequence for $\widehat{Z}(K_I; (D^1, S^0))$ is generated by

$$y_\sigma = y_1 \otimes \cdots \otimes y_m$$

where σ is a simplex in K_I and

$$y_i = \begin{cases} s_i & \text{if } i \in \sigma \\ t_i & \text{if } i \notin \sigma \text{ and } i \in I \\ 1 & \text{if } i \notin I. \end{cases}$$

Where 1 is the multiplicative unit.

Define a cochain complex, C_{K_I} freely generated over \mathbb{Z} by $\{y_\sigma | \sigma \in K_I\}$ with differential

$$d(y_\sigma) = \sum_{\tau} (-1)^{n(\tau)} y_\tau$$

where, for σ an n -simplex, the sum is over all $n+1$ simplices τ such that $\sigma \subset \tau \in K_I$. The integer $n(\tau)$ is defined by the usual graded sign convention for a derivation. i.e. there is a differential δ acting on each factor by $\delta(t_i) = s_i$, $\delta(s_i) = \delta(1) = 0$. d is defined on y_σ by extending δ to y_σ by the graded Leibniz rule.

Define C_K by

$$(7.4) \quad C_K = \bigoplus_{I \subset [m]} C_{K_I}.$$

The following is proven in [6].

Proposition 7.1. *There is an isomorphism of groups*

$$H^*(C_K) = H^*(Z(K; (D^1, S^0)))$$

Proof. C_K is the dual of the standard chain complex for $\bigvee_I \Sigma |K_I|$. The proposition now follows from Theorem 2.7. \square

The left lexicographical ordering of the simplices of K induce a filtration on $C_K = \bigoplus_I \mathbb{Z}\{y_\sigma | \sigma \in K_I\}$. The spectral sequence associated to this filtration is easily seen to be the spectral sequence $E_r(Z(K; (D^1, S^0)))$.

In [6] Cai gives a non-commutative product on the chain complex (7.4) which induces the cup product in $H^*(Z(K; (D^1, S^0)))$. This product specializes to the $*$ -product of Theorem 6.1.

Following [6] we define a non commutative product on C_K by extending the following product on the classes t_i and s_i :

$$(7.5) \quad t_i t_i = t_i, \quad t_i s_i = 0, \quad s_i t_i = s_i, \quad s_i s_i = 0$$

to C_K by the graded shuffle.

The product (7.5) induces the $*$ -product, (7.2), on the cochain complexes for N_{J_1} and N_{J_2} as follows. We suppose there are simplices $\alpha \in N_{J_1}$ and $\beta \in N_{J_2}$. There are the generators y_α and y_β of the cochains of N_{J_1} and N_{J_2} respectively. The graded shuffle product of y_α and y_β followed by the coordinate wise product defined in (7.5) determine a signed monomial, y_γ in the t_i 's and s_i 's. Here $\gamma = \{i | s_i \text{ appears in the } i - \text{th coordinate of the monomial}\}$. The $*$ -product, (7.2) on cochains is given by

$$y_\alpha \otimes y_\beta \mapsto \pm y_\gamma \mapsto \begin{cases} \pm y_\gamma & \text{if } \gamma \cup \sigma_1 \cup \sigma_2 \in K \\ 0 & \text{otherwise} \end{cases}$$

The map (7.3) is induced by the map of cochains dual to the inclusion. Namely if y_γ is a generator of the cochains of $N_{J_1 \cup J_2}$ then $\iota^*(y_\gamma)$ is given by

$$y_\gamma \mapsto \begin{cases} y_\gamma & \text{if } \gamma \in N_{J'} \\ 0 & \text{otherwise} \end{cases}$$

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